Two phase compressible and immiscible flow in porous media: mathematical and numerical analysis

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TITRE

Ecoulement diphasique compressible et immiscible en milieu poreux : analyse mathématique et numérique

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Résumé

L'objectif de cette thèse est l'étude du problème de Cauchy pour les solutions faibles de trois problèmes (systèmes paraboliques dégénérés et fortement couplés) modélisant des écoulements diphasiques et compressibles en milieu poreux. La motivation de ce travail est un "benchmark" du GNR MoMaS pour l'étude de l'impact de l'écoulement du gaz dû à la corrosion des matériaux ferreux dans un site de stockage de déchets radioactifs. Cette thèse est divisée en trois chapitres indépendants.
Premièrement, on s'intéresse à l'analyse mathématique d'un problème modélisant l'écoulement de deux phases immiscibles et en considérant qu'une phase est compressible et l'autre est incompressible (eau/gaz). Deuxièmement, on traite le cas général du déplacement de deux fluides compressibles et immiscibles dans un milieu poreux. Enfin, le dernier chapitre est consacré à la construction et à la convergence de la méthode des volumes finis pour le système eau-gaz sous l'hypothèse que la densité du gaz est une fonction de la pression globale.

Mots clés

Ecoulement en milieu poreux, compressible, immiscible, volumes finis, systèmes paraboliques dégénérés, systèmes elliptiques, systèmes non linéaires, méthode de semi-discrétisation.

Abstract

The aim of this thesis is the study of the Cauchy problem (existence of weak solutions) for three degenerate highly coupled parabolic systems modeling compressible immiscible flow in porous media. The motivation of this work is a benchmark of the GNR MoMaS, to study the impact of the gas flow due to the corrosion of ferrous materials in a radioactive waste storage site. This thesis is divided into three independent chapters.

Firstly, we look at a problem modeling the flow of two immiscible phases and considering one phase is compressible and the other is incompressible (water/gas). Secondly, we consider the problem modeling two-compressible immiscible flow in porous media. An existence results for both problems established by a semi-discretization method. Finally, The fourth chapter is devoted to the construction and convergence of a multidimensional finite volume method (upwind scheme) for the gas-water model under the assumption that the gas density is a function of a global pressure.

Key words

porous medium, compressible, immiscible, finite volume, parabolic degenerate systems, elliptic systems, degenerate systems, nonlinear coupled systems, semi-discretization method.

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CHAPITRE 1

Introduction

1 Contexte général et scientifique

La modélisation des écoulements multiphasiques en milieu poreux prend une place importante en ingénierie pétrolière, par exemple la récupération des hydrocarbures, et des problèmes liés à la pollution de l'environnement. En effet, en ingénierie pétrolière, la technique de récupération secondaire du pétrole est largement utilisée, elle consiste à injecter de l'eau dans des puits réservés à cet effet (puits d'injection) afin de déplacer les hydrocarbures présents dans le gisement vers les puits de production. Il est alors naturel de considérer deux ou trois phases (eau, gaz, huile) afin de simuler l'écoulement dans ces gisements.

Les gisements pétroliers peuvent être utilisés pour la séquestration du CO2. Les émissions du CO2 ont fortement augmenté au cours des récentes années, entraînant une croissance de la teneur en CO2 dans l'atmosphère. Ce type de gaz à effets de serre serait responsable de la tendance du réchauffement climatique. Une façon de réduire la teneur en CO2 de l'atmosphère est de capturer le CO2 émis afin de le séquestrer dans des sites de stockage. Plusieurs options de stockage sont envisagées : stockage dans des gisements d'hydrocarbures déplétés, veines de charbons inexploitées et aquifères salins profonds. La capacité potentielle de stockage du CO2 dans des gisements et dans des aquifères profonds est à la mesure des quantités de CO2 émises. Il est clair que les gisements pétroliers sont un bon moyen pour la séquestration du CO2. Actuellement, le CO2 est utilisé dans certains gisements pour le récupération secondaire du pétrole. Ce qui conduit naturellement à l'étude des écoulements diphasiques compressibles.

L'objet de cette thèse est essentiellement de simuler numériquement et d'analyser mathématiquement les écoulements de type eau-gaz dans un milieu poreux.

Ce travail est motivé par le Benchmark Couplex–Gaz 2 proposé par l'ANDRA lors des rencontres du GDR MoMaS sur l'étude de l'impact de l'écoulement du gaz dû à la corrosion des matériaux ferreux dans un site de stockage des déchets radioactifs. En effet, une quantité importante d'hydrogène produite par corrosion des colis de stockage entraîne une augmentation significative de la pression d'hydrogène autour des alvéoles des déchets. Une telle surpression risque d'endommager les colis de stockage, les matériaux de confinement des déchets et de fracturer le milieu géologique.
Le modèle physique complet est un problème biphasique (eau/gaz) tenant compte de l'hydrogène sous forme gazeuse et dissoute dans l'eau.

On s'intéresse au déplacement des fluides dans un milieu poreux (gisement pétrolier, site de stockage) constitué d'un seul type de roche caractérisé par la porosité, le tenseur des perméabilités intrinsèques, les pressions capillaires et les perméabilités relatives. Le fluide est constitué de deux phases compressibles ou compressible/incompressible, immiscibles et sans interaction chimique entre elles. Ici, la méthode de récupération secondaire du pétrole est modélisée. Elle consiste à injecter un fluide dans des puits d'injection (l'eau ou le CO2) afin de déplacer les hydrocarbures vers les puits de production.

Dans [47], les auteurs s'intéressent à l'analyse mathématique des écoulements diphasiques immiscibles compressibles en milieu poreux, notamment aux écoulements eau–gaz sans dissolution. Sous l'hypothèse que la densité du gaz dépend de la pression globale (la phase eau est considérée incompressible), l'existence de solutions pour le problème dégénéré est prouvée. Dans [45], le cas de deux fluides compressibles, immiscibles et sous l'hypothèse que les densités dépendent de la pression globale a été étudié. Cette hypothèse est justifiée dans les travaux de J. Jaffré et C. Chavent [1] lorsque la variation des densités par rapport à la pression capillaire est faible.

1. *Mathematical models and finite elements for reservoir simulation. Single phase, multiphase and multicomponent flows through porous media,* Studies in Mathematics and its Applications ; 17, North-Holland Publishing Comp., 1986.

Ici, on traite les modèles complets en supposant que la densité de chaque phase dépend de sa propre pression et en généralisant l'analyse aux écoulements multiphasiques. En effet, des nouvelles estimations d'énergies sont obtenues pour contrôler les vitesses de chaque phase et ensuite les termes capillaires. Cette nouvelle approche ne nécessite pas la formulation du problème diphasique compressible et immiscible en fonction de la pression globale. Par contre, la notion de la pression globale est introduite pour obtenir un résultat de compacité. Cette analyse mathématique nous conduit naturellement au développement de schémas numériques pour la simulation des écoulements eau/gaz. En effet, un schéma aux volumes finis en dimension 2 et 3 d'espace pour simuler un écoulement eau-gaz en supposant que la phase gaz est compressible et celle de l'eau est incompressible est étudié. L'idée est de proposer un schéma numérique conservant les estimations d'énergies sur les solutions discrètes. Le schéma proposé est un schéma implicite construit sur un *maillage admissible* au sens de Eymard, Gallouët, Herbin [2] et en décentrant les mobilités selon le gradient de la pression globale aux interfaces des mailles. Un point important dans la construction de ce schéma est de considérer une approximation de la densité du gaz aux interfaces des mailles comme étant la moyenne le long des pressions entre les deux mailles voisines. Cette approximation a permis d'assurer des estimations a priori sur les solutions discrètes et assurer la convergence du schéma numérique.

Dans la suite, nous allons décrire les principaux modèles traités et décrire les principaux résultats de cette thèse.

2 Formulation mathématique

2.1 Deux fluides compressibles et immiscibles en milieu poreux

Les équations décrivant les déplacements de deux fluides immiscibles et compressibles sont données par la conservation de la masse de chaque phase. Le modèle est obtenu à partir de la loi de conservation de la masse, de la loi de Darcy et de la loi de la pression capillaire. Pour plus de détails sur ce type de modèles, on peut citer [44, 45, 47]

2. *Finite Volume Methods. Handbook of Numerical Analysis*, Vol. VII, P. Ciarlet, J.-L. Lions, eds.,North-Holland, (2000)

Conservation de la masse de chaque phase

$$\phi(x)\partial_t(\rho_i(p_i)s_i) + \mathrm{div}(\rho_i(p_i)\mathbf{V}_i) + \rho_i(p_i)s_i f_P(t,x) = \rho_i(p_i)s_i^I f_I(t,x) \tag{2.1}$$

où ϕ est la porosité du milieu, la porosité indique la proportion de fluide pouvant imprégner une roche :

$$\Phi = \frac{\text{Volume de pores(vide)}}{\text{Volume total}};$$

ρ_i est la densité du fluide i. On appellera fluide 1 et fluide 2 respectivement le fluide non mouillant et le fluide mouillant et s_1 et s_2 leurs saturations respectives.

La vitesse de chaque phase $\mathbf{V}_i$ est donnée par la *loi de Darcy* :

$$\mathbf{V}_i = -\mathbf{K}\frac{k_i(s_i)}{\mu_i}\Big(\nabla p_i - \rho_i(p_i)\mathbf{g}\Big), \qquad i = 1, 2. \tag{2.2}$$

où $\mathbf{K}$ est le tenseur de perméabilité du milieu poreux (la perméabilité intrinsèque K traduit la résistance exercée par la roche à l'écoulement, elle dépend de la nature des matériaux en présence et de la répartition géométrique des pores dans le milieu),
k_i est la perméabilité relative de la phase i, elle traduit le fait que plus la phase est présente dans le milieu plus elle est mobile. Noter aussi que la phase est immobile dès qu'elle absente, ceci implique que les perméabilités relatives satisfont

$$k_r(s_i = 0) = 0, \tag{2.3}$$

la fonction $s_i \mapsto k_i(s_i)$ est croissante (voir figures 1.1–1.2),
μ_i la viscosité (constante), $\mathbf{g}$ est la gravité, et p_i la pression de la phase i.

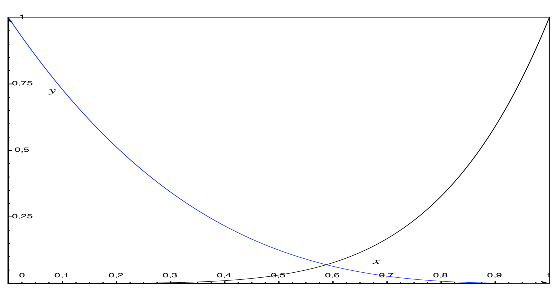

FIGURE 1.1 – $k_1(s_1)$ perméabilité de la phase 1(croissante) $k_2(s_1)$ perméabilité de la phase 2 (décroissante)

Dans les équations (2.1), l'injection et la récupération des fluides dans le milieu sont modélisées par les termes f_I et f_p. Les fonctions f_I et f_p sont respectivement les termes

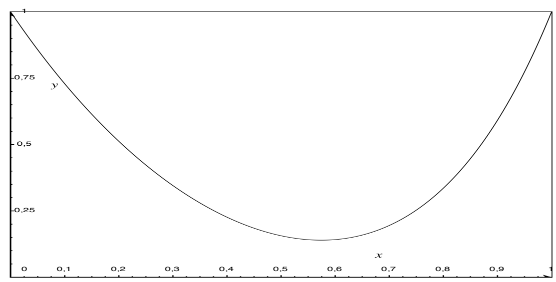

FIGURE 1.2 – Mobilité totale : $M = M_1 + M_2$

d'injection et production. Par ailleurs, les saturations des fluides injectés sont connues et sont notées s_i^I dans l'équation (2.1). Par définition de la saturation, on a

$$s_1(t,x) + s_2(t,x) = 1. \tag{2.4}$$

La courbure de la surface de contact entre deux fluides entraine une différence de pression appelé pression capillaire. Les expériences ont montré que les pressions capillaires sont fonctions de saturations. Ainsi, la pression capillaire ne dépend que de la saturation,

$$p_c(s_1(t,x)) := f(s_1(t,x)) = p_1(t,x) - p_2(t,x) \tag{2.5}$$

et est une fonction monotone croissante de la saturation, ($\frac{df}{ds}(s_1) \geq 0$, pour tout $s_1 \in [0,1]$).
Les inconnues de ce problème sont les saturations s_i, $i = 1,2$ et les pressions p_i, $i = 1,2$. Les relations (2.4) et (2.5) permettent de réduire les variables indépendantes. En effet, il suffit de connaître soit une saturation et une pression soit les deux pressions.

Nous verrons dans le chapitre 3, nous considérons les deux pressions comme inconnues.

Pour décrire le contexte physique, à savoir la récupération secondaire du pétrole, on complète le système par des conditions aux limites et des conditions initiales.
On désigne par Ω un ouvert borné de $\mathbb{R}^d$ ($d \geq 1$), de frontière $\partial\Omega$ régulière. Soit $\mathbf{n}$ le vecteur normal extérieur à $\partial\Omega$ et $[0,T]$ l'intervalle du temps d'étude. On note $Q_T = (0,T) \times \Omega$ et $\Sigma_T = (0,T) \times \partial\Omega$. La frontière $\partial\Omega$ est partitionnée comme suit :

$$\partial\Omega = \Gamma_1 \cup \Gamma_{imp} \qquad\qquad \Gamma_1 \cap \Gamma_{imp} = \emptyset$$

avec

Γ_1	frontière d'injection,
Γ_{imp}	frontière imperméable.

Conditions aux limites. On distingue, selon la nature des frontières considérées, les conditions aux limites suivantes :

▷ Sur Γ_1, partie de la frontière par laquelle l'eau (ou gaz) est injectée on impose

$$p_1(t,x) = 0, \quad p_2(t,x) = 0.$$

▷ Sur Γ_{imp}, frontière imperméable, on impose

$$\mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0.$$

Conditions initiales. Les conditions initiales, à l'instant $t = 0$, sont définies sur les pressions

$$\begin{aligned} p_1(0,x) &= p_1^0(x) \text{ dans } \Omega, \\ p_2(0,x) &= p_2^0(x) \text{ dans } \Omega. \end{aligned} \tag{2.6}$$

Le chapitre 2 est consacré à l'étude de ce système dans le cas où est l'une des phases est incompressible, modèle eau-gaz. Le chapitre 3 est consacré à l'étude de ce système dans le cas où les deux phases sont compressibles.

Formulation en saturation-pression globale.

Les équations (2.1) sont dégénérées à cause de la propriété physique (2.3), en effet dans la région où l'une des phases est absente alors la perméabilité est nulle et par conséquent la vitesse de la phase est nulle également, ainsi on perd le contrôle du gradient de la pression de cette phase. Pour se remédier à cela, C. Chavent et al. [22], ont introduit la notion de pression globale. La pression globale peut être contrôlée indépendamment de la présence des phases.

Pression globale. La pression p_i de chaque phase se représente comme étant une variation de la pression globale de la manière suivante :

$$p = p_2 + \tilde{p}(s_1) = p_1 + \bar{p}(s_1)$$

telles que les fonctions $\tilde{p}(s_1)$ et $\bar{p}(s_1)$ sont définies comme suit :

$$\tilde{p}'(s_1) = \frac{M_1(s_1)}{M(s_1)} f'(s_1), \qquad \bar{p}'(s_1) = -\frac{M_2(s_2)}{M(s_1)} f'(s_1),$$

où

$$\begin{aligned} M_i(s) &= k_i(s)/\mu_i && \text{la mobilité de la phase } i, \\ M(s) &= M_1(s) + M_2(s) && \text{la mobilité totale.} \end{aligned}$$

Alors, la vitesse de Darcy de chaque phase peut s'écrire sous la forme :

$$\mathbf{V}_i = -\mathbf{K}M_i(s_i)\nabla p - \mathbf{K}\alpha(s_1)\nabla s_i + \mathbf{K}M_i(s_i)\rho_i(p_i)\mathbf{g}. \tag{2.7}$$

où le terme capillaire

$$\alpha(s_1) = \frac{M_1(s_1)M_2(s_2)}{M(s_1)} \frac{df}{ds}(s_1) \geq 0.$$

Sur la figure 1.3, on montre l'allure de la fonction α.

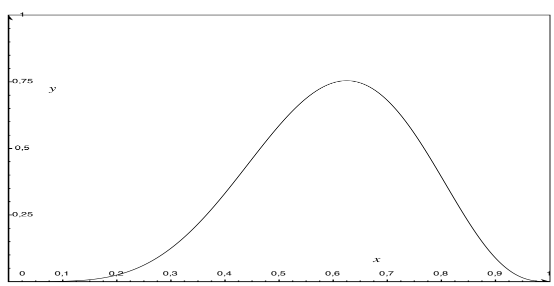

FIGURE 1.3 – Allure de la fonction α.

Dans le cas incompressible ($\rho_i = \textit{constante}$), il est facile de voir qu'en sommant les équations (2.1), on obtient une équation elliptique en pression

$$\mathrm{div}(V_1 + V_2) = -\,\mathrm{div}\Big(\mathbf{K}M(s_1)\nabla p - \mathbf{K}(M_1(s_1)\rho_1 + M_2(s_2)\rho_2)\mathbf{g})\Big) = 0,$$

et une équation parabolique en saturation

$$\partial_t s_1 - \mathrm{div}\Big(\mathbf{K}M_1(s_1)\nabla p + \mathbf{K}M_1(s_1)\rho_1\mathbf{g}\Big) = 0.$$

Le contrôle de la pression globale de l'équation elliptique permet ensuite le contrôle de la saturation.

Dans le cas compressible, nous n'allons pas exhiber une équation pour la pression et une équation pour la saturation. On traite le système de conservation de la masse dans sa formulation originale, la compressibilité complique évidemment l'analyse, on établit des nouvelles estimations d'énergies permettant le contrôle de la vitesse de chaque phase.

Dans [22, 44, 45, 47], l'hypothèse essentielle, classiquement formulée, est de considérer les densités des fluides comme une fonction de la pression globale :

$$\rho_i(p_i) = \rho_i(p). \tag{2.8}$$

En effet, selon Chavent et al. ([22], chapitre 3) la densité varie peu selon la pression capillaire. Sous l'hypothèse (2.8), les équations (2.1) s'écrivent

$$\phi(x)\partial_t(\rho_i(p)s_i) + \operatorname{div}(\rho_i(p)\mathbf{V}_i) + \rho_i(p)s_i f_P(t,x) = \rho_i(p)s_i^I f_I(t,x) \tag{2.9}$$

La vitesse de chaque phase :

$$\mathbf{V}_i = -\mathbf{K}M_i(s_i)\nabla p - \mathbf{K}\alpha(s_1)\nabla s_i + \mathbf{K}M_i(s_i)\rho_i(p)\mathbf{g}. \tag{2.10}$$

Pour clore le système, et par définition de la saturation on a :

$$s_1 + s_2 = 1. \tag{2.11}$$

A ce système, on ajoute les conditions suivantes :
Conditions aux limites
▹ Sur Γ_1, partie de la frontière par laquelle l'eau (ou gaz) est injectée on impose

$$s_1(t,x) = 0, \quad p(t,x) = 0$$

▹ Sur Γ_{imp}, frontière imperméable, on impose

$$\mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0.$$

Conditions initiales Les conditions initiales, a l'instant $t = 0$, définies sur les pressions

$$\begin{aligned} p(0,x) &= p_1^0(x) \text{ in } \Omega \\ s_1(0,x) &= p_2^0(x) \text{ in } \Omega. \end{aligned} \tag{2.12}$$

Dans le chapitre 4, on s'intéresse à la construction et à la convergence d'un schéma de type volumes finis pour ce modèle, en dimension 2 ou 3 d'espace, dans le cas où l'une des phases est incompressible.

3 Plan du mémoire

On s'intéresse dans cette thèse à l'étude du problème de Cauchy pour les solutions faibles de trois problèmes modélisant des écoulements diphasiques, immiscibles et compressibles.

On décrit un système parabolique non linéaire modélisant le déplacement de deux fluides compressibles et immiscibles dans un milieu poreux. En dimension 3, l'étude du problème de Cauchy pour les solutions faibles de deux modèles diphasiques a été réalisée. Le premier modèle traite de deux phases compressibles, le deuxième traite d'une phase compressible et d'une phase incompressible (écoulement eau/gaz).

De nouvelles estimations d'énergies ont été obtenues afin d'établir l'existence de solutions. La pression globale n'est pas nécessaire pour formuler le problème mais elle est utile afin d'obtenir un résultat de compacité.

3.1 Chapitre 2 : Système non linéaire dégénéré modélisant les déplacements immiscibles eau-gaz en milieu poreux

On s'intéresse à un problème modélisant l'écoulement de deux phases immiscibles et en considérant qu'une phase est compressible et l'autre est incompressible. On considère qu'un seul fluide est injecté (i.e $s_1^I = 0$, $s_2^I = 1$), et un seul fluide incompressible (i.e $\rho_2(p_2) = \rho_2 \in \mathbb{R}^+$). Les équations (2.1)–(2.2) se réduisent alors à

$$\phi(x)\partial_t(\rho_1(p_1)s_1)(t,x) + \mathrm{div}(\rho_1(p_1)\mathbf{V}_1)(t,x) + \rho_1(p_1)s_1 f_P(t,x) = 0, \tag{3.13}$$

$$\phi(x)\partial_t s_2(t,x) + \mathrm{div}(\mathbf{V}_2)(t,x) + s_2 f_P(t,x) = f_I(t,x), \tag{3.14}$$

$$s_1 + s_2 = 1, \tag{3.15}$$

$$f(s_1) = p_1 - p_2. \tag{3.16}$$

La première équation est la conservation de la masse du gaz, La deuxième est la conservation de la masse du fluide incompressible -l'eau en général-, dont la densité, constante, a été simplifiée. On considère aussi f_I le débit d'injection et f_P celui de la production. La loi d'état considérée est une fonction croissante par rapport à la pression et est bornée (voir (H6)).
Le réservoir est toujours noté Ω, un ensemble ouvert borné de $\mathbb{R}^d$. On pose, $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$. Au système (3.13)-(3.14), on ajoute les conditions aux

limites suivantes, $\partial\Omega = \Gamma_1 \cup \Gamma_{imp}$, où Γ_1 designe la frontière d'injection d'eau et Γ_{imp} son complémentaire.

$$\begin{cases} p_1(t,x) = 0, \;\; p_2(t,x) = 0 & \text{on } \Gamma_1 \\ \mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0 & \text{on } \Gamma_{imp} \end{cases} \tag{3.17}$$

Cela signifie que la pression est imposée dans une zone d'injection du bord du réservoir. On complète le problème par les conditions intiales suivantes

$$\begin{cases} p_1(0,x) = p_1^0(x) \text{ in } \Omega \\ p_2(0,x) = p_2^0(x) \text{ in } \Omega. \end{cases} \tag{3.18}$$

Les hypothèses portant sur les mobilités, la porosité du milieu, le tenseur de perméabilité et autres grandeurs physiques, sont similaires à celles de la section précédente,

(H1) il existe deux constantes ϕ_0 et ϕ_1 dans $W^{1,\infty}(\Omega))$ telles que $0 < \phi_0 \leq \phi(x) \leq \phi_1$ p.p. $x \in \Omega$.

(H2) Le tenseur $\mathbf{K}$ appartient à $(W^{1,\infty}(\Omega))^{d\times d}$. De plus, il existe deux constantes strictement positives k_0 et k_∞ telles que

$$\|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \leq k_\infty \quad \text{et} \quad (\mathbf{K}(x)\xi,\xi) \geq k_0|\xi|^2 \text{ (pour tout} \xi \in \mathbb{R}^d, \text{ p.p. } x \in \Omega).$$

(H3) Les fonctions M_1 et M_2 appartiennent à $\mathcal{C}^0([0,1];\mathbb{R}^+)$ et $M_i(s_i = 0) = 0$. De plus, il existe une constante strictement positive m_0 telle que, pour tout $s_1 \in [0,1]$,

$$M_1(s_1) + M_2(s_2) \geq m_0.$$

(H4) La fonction $\alpha \in \mathcal{C}^0([0,1];\mathbb{R}^+)$ satisfait $\alpha(s_1) > 0$ pour $0 < s_1 \leq 1$, et $\alpha(0) = 0$. On définit $\beta(s_1) = \int_0^{s_1} \alpha(z)dz$, on suppose que β^{-1} est une fonction Hölderienne d'ordre θ, avec $0 < \theta \leq 1$, sur $[0,\beta(1)]$. Cela signifie qu'il existe une constante positive non nulle c telle que pour tout $s_1, s_2 \in [0,\beta(1)]$, on a $|\beta^{-1}(s_1) - \beta^{-1}(s_2)| \leq c|s_1 - s_2|^\theta$.

(H5) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \geq 0$ p.p. $(t,x) \in Q_T$

(H6) La densité ρ_1 est $\mathcal{C}^2(\mathbb{R})$, strictement croissante et il existe $\rho_m > 0$ et $\rho_M > 0$ telque $0 < \rho_m \leq \rho_1(p_1) \leq \rho_M$

(H7) La pession capillaire $f \in \mathcal{C}^0([0,1];\mathbb{R}^-)$, f est différentiable et $0 < \underline{f} \leq \frac{df}{ds}$.

On définit l'espace de Hilbert

$$H^1_{\Gamma_1} = \{u \in H^1(\Omega); u = 0 \text{ sur } \Gamma_1\}.$$

Theorem 1.1. *Sous les hypothèses (H1)–(H7), pour p_1^0, p_2^0 (défini par (3.18)) appartenant à $L^2(\Omega)$ et s_0 vérifiant $0 \leq s_0 \leq 1$ p.p. Ω, il existe (p_1, p_2) solution de (3.13),* *(3.14) vérifiant,*

$$p_i \in L^2(0,T;L^2(\Omega)),\ \phi\partial_t(\rho_i(p_i)s_i) \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))'),\ i=1,2, \tag{3.19}$$
$$0 \le s_i(t,x) \le 1\ \textit{a.e in } Q_T,\ i=1,2,\ \beta(s_1) \in L^2(0,T;H^1(\Omega)) \tag{3.20}$$

such that for all $\varphi,\ \xi \in C^1(0,T;H^1_{\Gamma_1}(\Omega))$ *avec* $\varphi(T)=\xi(T)=0$,

$$\begin{aligned}
&-\int_{Q_T} \phi\rho_1(p_1)s_1\partial_t\varphi\,dxdt - \int_\Omega \phi(x)\rho_1(p_1^0(x))s_1^0(x)\varphi(0,x)\,dx \\
&+\int_{Q_T} \mathbf{K}M_1(s_1)\rho_1(p_1)\nabla p_1\cdot\nabla\varphi\,dxdt - \int_{Q_T} \mathbf{K}M_1(s_1)\rho_1^2(p_1)\mathbf{g}\cdot\nabla\varphi\,dxdt \\
&+\int_{Q_T} \rho_1(p_1)s_1 f_P\varphi\,dxdt = 0,
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
&-\int_{Q_T} \phi s_2\partial_t\xi\,dxdt - \int_\Omega \phi(x)s_2^0(x)\xi(0,x)\,dx \\
&+\int_{Q_T} \mathbf{K}M_2(s_2)\nabla p_2\cdot\nabla\xi\,dxdt - \int_{Q_T} \mathbf{K}M_2(s_2)\rho_2\mathbf{g}\cdot\nabla\xi\,dxdt \\
&+\int_{Q_T} s_2 f_P\xi\,dxdt = \int_{Q_T} f_I\,\xi\,dxdt,
\end{aligned} \tag{3.22}$$

et enfin les conditions initiales sont satisfaites dans un sens faible comme suit :

$$\textit{Pour tout } \psi \in H^1_{\Gamma_1}(\Omega) \textit{ les fonctions } t \longrightarrow \int_\Omega \phi\rho_1(p_1(t,x))s_1(t,x)\psi(x)\,dx \in \mathcal{C}^0([0,T]), \tag{3.23}$$

$$\textit{et } t \longrightarrow \int_\Omega \phi s_2(t,x)\psi(x)\,dx \in \mathcal{C}^0([0,T]) \tag{3.24}$$

de plus, on a

$$\int_\Omega \phi\rho_1(p_1(0,x)s_1(0,x)\psi(x)\,dx = \int_\Omega \phi\rho_1(p_1^0)s_1^0\psi\,dx \tag{3.25}$$
$$\int_\Omega \phi s_2(0,x)\psi\,dx = \int_\Omega \phi s_2^0\psi\,dx. \tag{3.26}$$

Le point clef de ce théorème d'existence est d'obtenir une estimation L^2 sur ∇p et $\nabla\beta(s_1)$. Pour cela, on note

$$g_1(p_1) := \int_0^{p_1} \frac{1}{\rho_1(\xi)}\,d\xi, \tag{3.27}$$
$$\mathcal{H}_1(p_1) := \rho_1(p_1)g_1(p_1) - p_1, \tag{3.28}$$

où $\mathcal{H}_1'(p_1) = \rho_1'(p_1)g_1(p_1)$, $\mathcal{H}_1(0) = 0$, $\mathcal{H}_1(p_1) \geq 0$ pour tout p_1, et $\mathcal{H}_1$ est sous-linéaire.(i.e $|\mathcal{H}_1(p_1) \leq C|p_1|$).
On multiplie (3.13) par $g_1(p_1)$ et (3.14) par p_2 et on somme ces deux estimations. Après intégration en espace, il reste

$$\frac{d}{dt}\int_\Omega \phi\left(s_1\mathcal{H}_1(p_1) + \int_0^{s_1} f(\xi)\,d\xi\right)dx + \int_\Omega \mathbf{K}M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx$$
$$+\int_\Omega \mathbf{K}M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx - \int_\Omega \mathbf{K}M_1(s_1)\rho_1(p_1)\mathbf{g}\cdot\nabla p_1\,dx - \int_\Omega \mathbf{K}M_2(s_2)\rho_2\mathbf{g}\cdot\nabla p_2\,dx$$
$$+\int_\Omega \rho_1(p_1)s_1f_pg_1(p_1)\,dx + \int_\Omega s_2f_pp_2\,dx = \int_\Omega f_Ip_2\,dx.$$

En utilisant les hypothèses $(H3)$ et $(H6)$ ainsi que la borne de la fonction $\mathcal{H} \geq 0$ et $g_1(p_1)$ est sous-linéaire, on déduit

$$\int_{Q_T} M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx + \int_{Q_T} M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx \leq C. \tag{3.29}$$

On a

$$\nabla p = \nabla p_2 + \frac{M_1}{M}\nabla f(s_1) = \nabla p_1 - \frac{M_2}{M}\nabla f(s_1), \tag{3.30}$$

et par conséquent, on a de l'égalité principale

$$\int_{Q_T} M|\nabla p|^2\,dx + \int_{Q_T}\frac{M_1M_2}{M}|\nabla f(s_1)|^2\,dx = \int_{Q_T} M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx + \int_{Q_T} M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx. \tag{3.31}$$

L'hypothèse $(H3)$ assure alors que $p \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$ et $\beta(s_1) \in L^2(0,T;H^1(\Omega))$. L'aspect dégénéré en évolution sur la variable pression ne permet pas d'obtenir de compacité en variable pression p_1. Par contre, le terme d'évolution $\partial_t(\rho_1(p_1)s_1)$ dans l'équation (3.13) nous permet d'obtenir de la compacité en variable $\rho_1(p_1)s_1$. D'autres difficultés techniques apparaissent alors, en particulier l'identification de la limite de la variable $\rho_1(p_{1,h})s_{1,h}$ où $(p_{1,h}, s_{1,h})$ est solution d'un problème approché. Cette identification est rendue possible grâce à la monotonie de la fonction ρ_1, alors même que l'on ne dispose que de la convergence faible dans la variable pression et de la convergence forte sur la saturation dans L^2.
Le choix du problème approché doit dans un premier temps assurer la positivité de la saturation et ensuite pouvoir définir la pression dans un processus d'ajout de la dissipation artificielle. La preuve du théorème 1.1 s'effectue en deux étapes, la première consiste à prouver l'existence des solutions pour le problème non-dégénéré, la fonction

M_i (dégénérée en 0), on remplace dans l'quation (3.13) le terme dégénérée

$$-\operatorname{div}(\mathbf{K}\rho_1(p_1)M_1(s_1)\nabla p_1)$$

par un terme non dégénéré

$$-\operatorname{div}(\mathbf{K}\rho_1(p_1^\eta)M_1(s_1^\eta)\nabla p_1^\eta) - \eta\operatorname{div}(\rho_1(p_1^\eta)\nabla(p_1^\eta - p_2^\eta))$$

et nous remplaçons aussi dans l'quation (3.14) la terme dégénérée

$$-\operatorname{div}(\mathbf{K}M_2(s_2)\nabla p_2)$$

par un terme non dégénéré

$$-\operatorname{div}(\mathbf{K}M_2(s_2^\eta)\nabla p_2^\eta) - \eta\operatorname{div}(\nabla(p_2^\eta - p_1^\eta)).$$

L'existence des solutions du problème non dégénéré est basée sur une méthode de semi-discrétisation en temps ([3]). Soit $T > 0$, $N \in I\!N^*$ et $h = \frac{T}{N}$. On définit la suite paramétrée par h :

$$s_{1,h}^0(x) = s_1^0(x) \in [0,1], \text{ a.e. in } \Omega \tag{3.32}$$

$$p_{i,h}^0(x) = p_i^0(x) \text{ a.e. in } \Omega, \tag{3.33}$$

pour tout $n \in [0, N-1]$, soit $(p_{1,h}^n, (p_{2,h}^n) \in L^2(\Omega) \times L^2(\Omega)$ avec $0 \le s_{i,h}^n \le 1$, on désigne par $(f_P)_h^{n+1} = \frac{1}{h}\int_{nh}^{(n+1)h} f_P(\tau)\,d\tau$ et $(f_I)_h^{n+1} = \frac{1}{h}\int_{nh}^{(n+1)h} f_I(\tau)\,d\tau$ alors on définit, $(p_{1,h}^{n+1}, (p_{2,h}^{n+1}) \in H_{\Gamma_1}^1(\Omega) \times H_{\Gamma_1}^1(\Omega)$ avec $0 \le s_{i,h}^{n+1} \le 1$ solution de

$$\begin{aligned}\phi\partial_t(\rho_1(p_1^\eta)s_1^\eta) - \operatorname{div}(\mathbf{K}\rho_1(p_1^\eta)M_1(s_1^\eta)\nabla p_1^\eta) + \operatorname{div}(\mathbf{K}\rho_1^2(p_1^\eta)M_1(s_1^\eta)\mathbf{g})&\\ - \eta\operatorname{div}(\rho_1(p_1^\eta)\nabla(p_1^\eta - p_2^\eta)) + \rho_1(p_1^\eta)s_1^\eta f_P = 0,&\end{aligned} \tag{3.34}$$

$$\begin{aligned}\phi\partial_t(s_2^\eta) - \operatorname{div}(\mathbf{K}M_2(s_2^\eta)\nabla p_2^\eta) + \operatorname{div}(\mathbf{K}\rho_2 M_2(s_2^\eta)\mathbf{g})&\\ - \eta\operatorname{div}(\nabla(p_2^\eta - p_1^\eta)) + s_2^\eta f_P = f_I,&\end{aligned} \tag{3.35}$$

avec les conditions intiales (3.18) et les conditions aux limites

$$\begin{cases} p_1^\eta(t,x) = 0, \;\; p_2^\eta(t,x) = 0 & \text{on } (0,T)\times\Gamma_1 \\ \Big(\mathbf{K}M_1(s_1^\eta)(\nabla p_1^\eta - \rho_1(p_1^\eta)\mathbf{g}) + \eta\nabla(p_1^\eta - p_2^\eta)\Big)\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{imp} \\ \Big(\mathbf{K}M_2(s_2^\eta)(\nabla p_2^\eta - \rho_2\mathbf{g}) - \eta\nabla(p_1^\eta - p_2^\eta)\Big)\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{imp} \end{cases} \tag{3.36}$$

où $\mathbf{n}$ est la normale extérieure à la limite Γ_{imp}.

L'existence des solutions pour ce système elliptique est obtenue grâce au théorème de Leray–Schauder et en utilisant deux régularisations en pressions : la première consiste à projeter la pression sur les N premiers vecteurs propres de l'opérateur $-\Delta p$ dans les équations (3.54)(3.35) du chapitre 2 et la seconde consiste à ajouter de la dissipation artificielle par remplacer M_i par $M_i^\varepsilon = M_i + \varepsilon$. Ensuite une version discréte de l'inégalité (3.51) permet l'obtention des estimations uniformes sur les solutions indépendantes de h et donc le passage à la limite quand h tend vers zéro. Enfin, on s'intéresse au passage à la limite quand η tend vers zéro. Il est clair qu'il est inespéré de démontrer que $(\nabla s^\eta)_\eta$ est uniformément borné dans $L^2(Q_T)$, par contre on établit que les suites $(\beta(s_1^\eta))_\eta$ et $(p^\eta)_\eta$ sont uniformément bornées dans $L^2(0,T;H^1(\Omega))$ et $(\sqrt{\eta}\,\nabla f(s_1^\eta))_\eta$ et $(\sqrt{M_i(s_i^\eta)}\,\nabla p_i^\eta)_\eta$ sont uniformément bornées dans $L^2(Q_T)$. Ces estimations sont essentielles pour démontrer le théorème 1.1.

3.2 Chapitre 3 : Ecoulement diphasique compressible immiscible en milieu poreux

On considère le déplacement diphasique, immiscible d'un fluide compressible par un autre. La différence par rapport au modèle eau-gaz mélange traitée dans le premier chapitre est que :
- La densité du fluide considère comme une fonction de sa pression correspondant.

$$\rho_i = \rho_i(p_i) \qquad i = 1, 2.$$

- Les termes source injectés sont contraints par

$$s_1^I + s_2^I = 1 \qquad s_i^I \geq 0$$

- Le terme capillaire α est dégénéré en 0 et 1.
On considère la formulation à deux pressions

$$\begin{aligned}
&\phi(x)\partial_t(\rho_i(p_i)s_i)(t,x) + \operatorname{div}(\rho_i(p_i)\mathbf{V}_i)(t,x) \\
&\qquad + \rho_i(p_i)s_i f_P(t,x) = \rho_i(p_i)s_i^I f_I(t,x), \quad i = 1, 2.
\end{aligned} \tag{3.37}$$

$$s_1 + s_2 = 1, \tag{3.38}$$

$$f(s_1) = p_1 - p_2. \tag{3.39}$$

avec les conditions initiales et les conditions aux limites suivante :

Le réservoir est toujours noté Ω, un ensemble borné de $\mathbb{R}^d$. On pose, $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$. Au système (3.13)-(3.14), on ajoute les conditions aux limites suivantes, $\partial\Omega = \Gamma_1 \cup \Gamma_{imp}$, où Γ_1 désigne la frontière d'injection d'eau et Γ_{imp} son complémentaire.

$$\begin{cases} p_1(t,x) = 0, \;\; p_2(t,x) = 0 & \text{on } \Gamma_1 \\ \mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0 & \text{on } \Gamma_{imp} \end{cases} \tag{3.40}$$

Cela signifie que la pression est imposée dans une zone d'injection du bord du réservoir. On complète le problème par les conditions initiales suivantes

$$\begin{cases} p_1(0,x) = p_1^0(x) \text{ in } \Omega \\ p_2(0,x) = p_2^0(x) \text{ in } \Omega. \end{cases} \tag{3.41}$$

Les hypothèses portant sur les mobilités, la porosité du milieu, le tenseur de perméabilité et autres grandeurs physiques, sont similaires à celles de la section précédente,

(H1) il existe deux constantes ϕ_0 et ϕ_1 telles que $0 < \phi_0 \le \phi(x) \le \phi_1$ p.p. $x \in \Omega$.

(H2) Le tenseur $\mathbf{K}$ appartient à $(W^{1,\infty}(\Omega))^{d\times d}$. De plus, il existe deux constantes strictement positives k_0 et k_∞ telles que

$$\|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \le k_\infty \quad \text{et} \quad (\mathbf{K}(x)\xi,\xi) \ge k_0|\xi|^2 \text{ (for all} \xi \in \mathbb{R}^d, \text{ p.p. } x \in \Omega).$$

(H3) Les fonctions M_1 et M_2 appartiennent à $\mathcal{C}^0([0,1];\mathbb{R}^+)$ et $M_i(s_i = 0) = 0$. De plus, il existe une constante strictement positive m_0 telle que, pour tout $s_1 \in [0,1]$,

$$M_1(s_1) + M_2(s_2) \ge m_0.$$

(H4) La fonction $\alpha \in \mathcal{C}^0([0,1];\mathbb{R}^+)$ satisfait $\alpha(s_1) > 0$ pour $0 < s_1 < 1$, et $\alpha(0) = \alpha(1) = 0$.
On définit $\beta(s_1) = \int_0^{s_1} \alpha(z)dz$, on suppose que β^{-1} est une fonction Hölderienne d'ordre θ, avec $0 < \theta \le 1$, sur $[0,\beta(1)]$. Cela signifie qu'il existe une constante positive non nulle c telle que pour tout $s_1, s_2 \in [0,\beta(1)]$, on a $|\beta^{-1}(s_1) - \beta^{-1}(s_2)| \le c|s_1 - s_2|^\theta$.

(H5) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x),\ f_I(t,x) \ge 0$ p.p. $(t,x) \in Q_T$
$s_i^I(t,x) \ge 0$ $(i = 1,2)$ et $s_1^I(t,x) + s_2^I(t,x) = 1$ p.p. $(t,x) \in Q_T$.

(H6) La densité ρ_i $(i = 1,2.)$ est $\mathcal{C}^2(\mathbb{R})$, strictement croissante et il existe $\rho_m > 0$ et $\rho_M > 0$ telque $0 < \rho_m \le \rho_i(p_i) \le \rho_M$

(H7) La pession capillaire $f \in \mathcal{C}^0([0,1];\mathbb{R}^-)$, f est différentiable et $0 < \underline{f} \le \frac{df}{ds}$.

Theorem 1.2. *Sous les hypothèses (H1)–(H7), pour p_1^0, p_2^0 (défini par (3.41)) appartenant à $L^2(\Omega)$ et s_0 vérifiant $0 \le s_0 \le 1$ p.p. Ω, il existe (p_1,p_2) solution de (3.37)* *vérifiant,*

$$p_i \in L^2(0,T;L^2(\Omega)), \quad M_i(s_i)\nabla p_i \in L^2(0,T;L^2(\Omega)) \tag{3.42}$$

$$\phi\partial_t(\rho_i(p_i)s_i) \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))'),\ i=1,2, \tag{3.43}$$

$$0 \le s_i(t,x) \le 1 \ a.e\ in\ Q_T,\ i=1,2,\ \beta(s_1) \in L^2(0,T;H^1(\Omega)) \tag{3.44}$$

tel que pour tout $\varphi_i, \in C^1(0,T;H^1_{\Gamma_1}(\Omega))$ *avec* $\varphi(T)=\xi(T)=0$,

$$\begin{aligned}
&-\int_{Q_T} \phi\rho_i(p_i)s_i\partial_t\varphi_i\,dxdt - \int_\Omega \phi(x)\rho_i(p_i^0(x))s_i^0(x)\varphi_i(0,x)\,dx \\
&+\int_{Q_T} \mathbf{K}M_i(s_i)\rho_i(p_i)\nabla p_i\cdot\nabla\varphi_i\,dxdt - \int_{Q_T} \mathbf{K}M_i(s_i)\rho_i^2(p_i)\mathbf{g}\cdot\nabla\varphi_i\,dxdt \\
&+\int_{Q_T} \rho_i(p_i)s_i f_P\varphi_i\,dxdt = \int_{Q_T} \rho_i(p_i)s_i^I f_I\varphi_i\,dxdt,
\end{aligned} \tag{3.45}$$

et enfin les conditions initiales sont satisfaites au sens faible suivant :
pour $i=1,2$,

$$\text{pour tout } \psi \in H^1_{\Gamma_1}(\Omega) \text{ les fonctions } t \longrightarrow \int_\Omega \phi\rho_i(p_i(t,x))s_i(t,x)\psi(x)\,dx \in \mathcal{C}^0([0,T]), \tag{3.46}$$

de plus, on a

$$\Big(\int_\Omega \phi\rho_i(p_i)s_i\psi\,dx\Big)(0) = \int_\Omega \phi\rho_i(p_i^0)s_i^0\psi\,dx \tag{3.47}$$

Le point clef de ce théorème d'existence est d'obtenir une estimation L^2 sur ∇p et $\nabla\beta(s_1)$. Pour cela, on note

$$g_i(p_i) := \int_0^{p_i} \frac{1}{\rho_i(\xi)}\,d\xi, \tag{3.48}$$

$$\mathcal{H}_i(p_i) := \rho_i(p_i)g_i(p_i) - p_i, \tag{3.49}$$

où $\mathcal{H}_i'(p_i) = \rho_i'(p_i)g_i(p_i),\ \mathcal{H}_i(0)=0,\ \mathcal{H}_i(p_i)\ge 0$ pour tout p_i, et $\mathcal{H}_i$ est souslinéaire.(i.e $|\mathcal{H}_i(p_i) \le C|p_i|)$.
On multiplie (3.37) par $g_i(p_i)$ pour $i=1,2$ et on somme ces des estimations. Après

intégration en espace, il reste

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi\left(s_1\mathcal{H}_1(p_1)+s_2\mathcal{H}_2(p_2)+\int_0^{s_1} f(\xi)\,d\xi\right)dx\\
&+\int_\Omega \mathbf{K}M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx+\int_\Omega \mathbf{K}M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx\\
&-\int_\Omega \mathbf{K}M_1(s_1)\rho_1(p_1)\mathbf{g}\cdot\nabla p_1\,dx-\int_\Omega \mathbf{K}M_2(s_2)\rho_2(p_2)\mathbf{g}\cdot\nabla p_2\,dx\\
&+\int_\Omega \rho_1(p_1)s_1 f_p g_1(p_1)\,dx+\int_\Omega \rho_2(p_2)s_2 f_p g_2(p_2)\,dx\\
&=\int_\Omega \rho_1(p_1)s_1^I f_I g_1(p_1)\,dx+\int_\Omega \rho_2(p_2)s_2^I f_I g_2(p_2)\,dx.
\end{aligned}
\tag{3.50}
$$

Un point clé est d'obtenir le premier terme dans l'égalité ci-dessus. On note

$$
\begin{aligned}
D &= \partial_t(\rho_1(p_1)s_1)g_1(p_1)+\partial_t(\rho_2(p_2)s_2)g_2(p_2)\\
&= \partial_t(\rho_1(p_1)s_1g_1(p_1))+\partial_t(\rho_2(p_2)s_2g_2(p_2))-s_1\partial_t p_1-s_2\partial_t p_2.
\end{aligned}
$$

On a $s_1+s_2=1$, alors $s_1\partial_t p_1+s_2\partial_t p_2=s_1\partial_t f(s_1)+\partial_t p_2=\partial_t G(s_1)+\partial_t p_2$, où G est une primitive de $s_1 f'(s_1)$. On peut écrire D comme $D=\partial_t E$ où E est définie par

$$
\begin{aligned}
E &= \rho_1(p_1)s_1g_1(p_1)+\rho_2(p_2)s_2g_2(p_2)-G(s_1)-p_2\\
&= s_1(\rho_1(p_1)g_1(p_1)-p_1)+s_2(\rho_2(p_2)s_2g_2(p_2)-p_2)-G(s_1)+s_1f(s_1),
\end{aligned}
$$

De la définition des fonctions $\mathcal{H}_i$ $(i=1,2)$ et G, l'expression de E est équivalente à :

$$
E = s_1\mathcal{H}_1(p_1)+s_2\mathcal{H}_2(p_2)+\int_0^{s_1} f(\xi)\,d\xi.
$$

En utilisant les hypothèses ($H2$) et ($H5$) ainsi que la borne de la fonction $\mathcal{H}\geq 0$ et g_1"(p_1) est sous-linéaire, on déduit

$$
\int_{Q_T} M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx+\int_{Q_T} M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx\leq C.
\tag{3.51}
$$

On a

$$
\nabla p=\nabla p_2+\frac{M_1}{M}\nabla f(s_1)=\nabla p_1-\frac{M_2}{M}\nabla f(s_1),
\tag{3.52}
$$

et donc

$$
\begin{aligned}
&\int_{Q_T} M|\nabla p|^2\,dx+\int_{Q_T}\frac{M_1M_2}{M}|\nabla f(s_1)|^2\,dx=\\
&\qquad\int_{Q_T} M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx+\int_{Q_T} M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx.
\end{aligned}
\tag{3.53}
$$

L'hypothèse ($H2$) assure alors que $p \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$ et $\beta(s_1) \in L^2(0,T;H^1(\Omega))$.

Avant d'établir le théorème 1.2, on montre l'existence de solution du système (3.37) sous l'hypothèse ($H1$)–($H7$), en ajoutant un terme de dissipation en saturation dans chaque équation (problème non dégénéré). Pour cela, la méthode de semi discrétisation en temps est employée. A chaque intervalle en temps, on établit d'abord l'existence des solutions du problème elliptique. Comme dans le chapitre précédent, différentes régularisations sont alors introduites pour mener l'existence d'un point fixe via le théorème de Leray-Schauder. Enfin, on établit un lemme de compacité permettant le passage du problème non dégénéré au problème initial. On considère le système non dégénéré paramétré par η :

$$\begin{aligned}\phi\partial_t(\rho_i(p_i^\eta)s_i^\eta) - \operatorname{div}(\mathbf{K}\rho_i(p_1^\eta)M_i(s_i^\eta)\nabla p_i^\eta) + \operatorname{div}(\mathbf{K}\rho_i^2(p_1^\eta)M_i(s_i^\eta)\mathbf{g})\\ + (-1)^i\eta\operatorname{div}(\rho_i(p_i^\eta)\nabla(p_1^\eta - p_2^\eta)) + \rho_i(p_i^\eta)s_i^\eta f_P = \rho_i(p_i^\eta)s_i^I f_I, \end{aligned} \tag{3.54}$$

avec les condition initiales (3.41), et les conditions aux limites suivantes

$$\begin{cases} p_1^\eta(t,x) = 0, \;\; p_2^\eta(t,x) = 0 & \text{on } \Gamma_1 \\ \Big(-\mathbf{K}M_i(s_i^\eta)(\nabla p_i^\eta - \rho_1(p_i^\eta)\mathbf{g}) + (-1)^i\eta\nabla(p_1^\eta - p_2^\eta)\Big)\cdot\mathbf{n} = 0 & \text{on } \Gamma_{imp} \end{cases} \tag{3.55}$$

où $\mathbf{n}$ est le vecteur normal sortant de Γ_{imp}.

Enfin, on s'intéresse au passage à la limite quand η tend vers zéro. On établit que les suites $(\beta(s_1^\eta))_\eta$ et $(p^\eta)_\eta$ sont uniformément bornées dans $L^2(0,T;H^1(\Omega))$ et $(\sqrt{\eta}\,\nabla f(s_1^\eta))_\eta$ et $(\sqrt{M_i(s_i^\eta)}\,\nabla p_i^\eta)_\eta$ sont uniformément bornées dans $L^2(Q_T)$. Enfin, pour passer à la limite sur η, on établit un résultat de compacité sur les solutions (p_1,p_2) et en utilisant le fait que l'application $\mathbb{H} : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R} \times [0,\beta(1)]$ definie par

$$\mathbb{H}(\rho_1(p_1)s_1, \rho_2(p_2)s_2) = (p, \beta(s_1)) \tag{3.56}$$

est un homéomorphisme.

3.3 Chapitre 4 : Convergence d'un schéma de volumes finis pour le modèle eau-gaz

On s'intéresse dans ce chapitre à la construction et à la convergence de la méthode des volumes finis pour le système eau-gaz sous l'hypothèse que la densité du gaz est une fonction de la pression globale.

Le système (2.1) se réduit dans ce cas à :

$$\phi\partial_t(\rho(p)s) - \operatorname{div}(\mathbf{K}\rho(p)M_1(s)\nabla p) - \operatorname{div}(\mathbf{K}\rho(p)\alpha(s)\nabla s) \\ + \operatorname{div}(\mathbf{K}\rho^2(p)M_1(s)\mathbf{g}) + \rho(p)sf_P = 0, \quad (3.57)$$

$$\phi\partial_t s + \operatorname{div}(\mathbf{K}M_2(s)\nabla p) - \operatorname{div}(\mathbf{K}\alpha(s)\nabla s) + \operatorname{div}(\mathbf{K}\rho_2 M_2(s)\mathbf{g}) + sf_P = f_P - f_I. \quad (3.58)$$

La première équation est la conservation de la masse du gaz, la deuxième est la conservation de la masse du fluide incompressible -l'eau en général-, dont la densité, constante, a été simplifiée.
Le réservoir est toujours noté Ω, un ensemble ouvert borné de $\mathbb{R}^N$. On pose, $Q_T = (0,T)\times\Omega$, $\Sigma_T = (0,T)\times\partial\Omega$. Au système (3.57)-(3.58), on ajoute les conditions aux limites suivantes, $\partial\Omega = \Gamma_w \cup \Gamma_i$, où Γ_w désigne la frontière d'injection d'eau et Γ_i son complémentaire.

$$\begin{cases} s(t,x) = 0, \quad p(t,x) = 0 \text{ on } \Gamma_w \\ \mathbf{V}_1\cdot\mathbf{n} = \mathbf{V}_2\cdot\mathbf{n} = 0 \text{ on } \Gamma_i, \end{cases} \quad (3.59)$$

On complète le problème par les conditions initiales suivantes

$$\begin{cases} s(0,x) = s_0(x), \text{ in } \Omega \\ p(0,x) = p_0(x) \text{ in } \Omega \end{cases} \quad (3.60)$$

Le schéma de volumes finis qu'on propose est valable pour un tenseur de perméabilité de type :

$$\mathbf{K} = k\,\mathcal{I}_d$$

où k est une constante positive. Quitte à faire un changement d'échelle en temps, on pose $k = 1$. Nous verrons comment généraliser au cas où la fonction k dépend de l'espace.

Ensuite, les hypothèses portant sur les mobilités, la porosité du milieu et autres grandeurs physiques, sont similaires à celles de la section précédente.

(H1) $\exists\ \phi_0$ et ϕ_1 dans $L^\infty(\Omega))$ telles que $0 < \phi_0 \le \phi(x) \le \phi_1$ p.p. $x \in \Omega$.

(H2) Les fonctions M_1 et $M_2 \in \mathcal{C}^0([0,1];\mathbb{R}^+)$ et $M_1(0) = 0$. De plus, il existe $m_0 > 0$ tel que

$$M(s) = M_1(s) + M_2(s) \ge m_0, \qquad s \in [0,1].$$

(H3) La fonction $\alpha \in \mathcal{C}^0([0,1];\mathbb{R}^+)$ satisfait $\alpha(s) > 0$ pour $0 < s \le 1$, et $\alpha(0) = 0$. On définit $\beta(s) = \int_0^s \alpha(z)dz$, on suppose que β^{-1} est une fonction Höldérienne d'ordre θ, avec $0 < \theta \leq 1$, sur $[0, \beta(1)]$.

(H4) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \geq 0$ p.p. $(t,x) \in Q_T$

(H5) La densité ρ est $\mathcal{C}^1(\mathbb{R})$, ρ est strictement croissante et il existe $\rho_m > 0$ et $\rho_M > 0$ tel que $0 < \rho_m \leq \rho_1(p_1) \leq \rho_M$

Definition 1.1. *Sous les hypothèses (H1)-(H5), et pour des données initiales (3.60) $p_0 \in L^2(\Omega)$ et s_0 satisfaisant $0 \leq s_0 \leq 1$ p.p. $x \in \Omega$. Le couple (s,p) est dit solution faible de (3.57)-(3.58) si*

$$0 \leq s \leq 1 \text{ a.e. in } Q_T, \quad \beta(s) \in L^2(0,T; H^1_{\Gamma_w}(\Omega)), \quad p \in L^2(0,T; H^1_{\Gamma_w}(\Omega)),$$

pour tout $\varphi, \xi \in \mathcal{D}\Big([0,T) \times \Omega\Big)$,

$$\begin{aligned} -\int_{Q_T} \phi\rho(p)s\partial_t\varphi\,dxdt - \int_\Omega \phi(x)u_0(x)\varphi(0,x)\,dx \\ + \int_{Q_T} \rho(p)M_1(s)\nabla p\cdot\nabla\varphi\,dxdt + \int_{Q_T}\rho(p)\nabla\beta(s)\cdot\nabla\varphi\,dxdt \\ - \int_{Q_T}\rho^2(p)M_1(s)\mathbf{g}\cdot\nabla\varphi\,dxdt + \int_{Q_T}\rho(p)sf_P\varphi\,dxdt = 0, \end{aligned} \tag{3.61}$$

$$\begin{aligned} -\int_{Q_T}\phi s\partial_t\xi\,dxdt - \int_\Omega \phi s_0(x)\xi(0,x)\,dx + \int_{Q_T}\nabla\beta(s)\cdot\nabla\xi\,dxdt \\ - \int_{Q_T} M_2(s)\nabla p\cdot\nabla\xi\,dxdt - \int_{Q_T}\rho_2 M_2(s)\mathbf{g}\cdot\nabla\xi\,dxdt \\ + \int_{Q_T} sf_P\xi\,dxdt = \int_{Q_T}(f_P - f_I)\xi\,dxdt. \end{aligned} \tag{3.62}$$

Schémas de volumes finis et résultats principaux

Soit $\mathcal{T}$ un maillage polygonal régulier et **admissible** (à préciser plus loin) du domaine Ω, constitué d'une famille de sous-domaines compacts, polygonaux, convexes, non vides K de Ω avec taille maximale (diamètre) h, et appelés volumes de contrôle.

Pour tout $K \in \mathcal{T}$, on note x_K le centre de K, $N(K)$ l'ensemble des voisins de K, $N_{\text{int}}(K)$ l'ensemble des voisins de K localisé à l'intérieur de $\mathcal{T}$, par $N_{\text{ext}}(K)$ l'ensemble des voisins de K sur la frontière $\partial\Omega$.
De plus, pour tout $L \in N_{int}(K)$ on note par $d_{K,L}$ le distance entre x_K et x_L, par $\sigma_{K,L}$ l'interface entre K et L, et par $\eta_{K,L}$ la normale unitaire à $\sigma_{K,L}$ orientée de K vers L. Et pour tout $\sigma \in N_{\text{ext}}(K)$, on note $d_{K,\sigma}$ le distance de x_K à σ. Donnons la figure 1.4 pour plus de clarté dans notre explication.

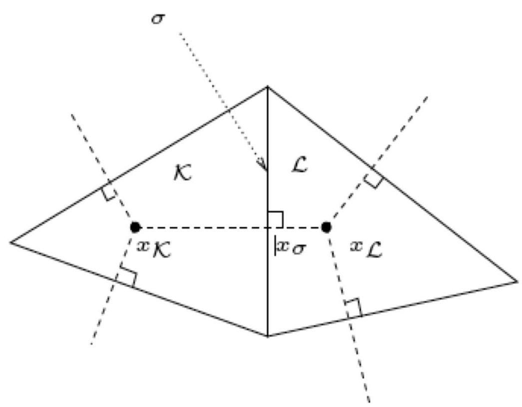

FIGURE 1.4 – Centre et distance

Pour tout $K \in \mathcal{T}$, $|K|$ désigne la mesure de Lebesgue dans $\mathbb{R}^d$ de K.
$\mathcal{T}$ est un **maillage orthogonal admissible** dans le sens de Eymard-Gallouët-Herbin [3].
L'admissibilité de $\mathcal{T}$ implique que $\overline{\Omega} = \cup_{K\in\mathcal{T}}\overline{K}$, et pour tout triangle $L \in N(K)$ (voisin de la maille K), le segment $[x_K x_L]$ est orthogonale à $\sigma_{K,L} = \overline{K} \cap \overline{L}$.
Un maillage conforme constitué de rectangles vérifie la condition d'orthogonalité, alors on peut prendre pour x_K le centre de gravité de K.
Pour un maillage conforme de triangles, on peut prendre pour x_K le centre du cercle circonscrit à K. Pour assurer que le centre x_K soit dans K, alors tous les angles du triangle doivent être aigus. On retrouve souvent cette condition en éléments finis pour assurer la monotonie du système linéaire.
L'avantage d'un tel maillage est de donner une approximation consistante de la dérivée dans la direction de la normale en utilisant uniquement deux points :

$$\nabla u(x) \cdot \eta_\sigma = \frac{u(x_L) - u(x_K)}{d_{K,L}} + \mathcal{O}(h), \forall x \in \sigma = \overline{K} \cap \overline{L},$$

on a également $x_L - x_K = d_{K,L}\eta_{K,L}$. Pour un vecteur $u_h = (u_K)_{K\in\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ donné sur

3. Finite Volume Methods. *Handbook of Numerical Analysis*, Vol. VII, P. Ciarlet, J.-L. Lions, eds., North-Holland, 2000.

$\mathcal{T}$, on associe la fonction constante par maille

$$u_h(x) = \sum_{K\in\mathcal{T}} u_K \mathbb{1}_K(x).$$

A partir de u_h, on définit $\nabla_h u_h$ le gradient discret constant par diamond $T_{K,L}$. On appelle **diamond** $T_{K,L}$, associé à l'arête $\sigma_{K,L}$, le polygone formé des quatre sommets x_L, x_K et les deux sommets de l'arête $\sigma_{K,L}$ (voir figure 1.5). On a alors le recouvrement suivant :

$$\Omega = \cup_{K\in\mathcal{T}} K = \cup_{\sigma\in\mathcal{E}} T_{K,L}.$$

On a aussi $|T_{K,L}| = \dfrac{1}{\ell}|\sigma_{K,L}| d_{K,L}$. Enfin, le gradient discret $\nabla_h u_h$ est défini constant par diamond $T_{K,L}$ comme suit

$$\nabla_h u_h(x) = \ell \frac{u_L - u_K}{d_{K,L}} \eta_{K,L} \text{ si } x \in T_{K,L}$$

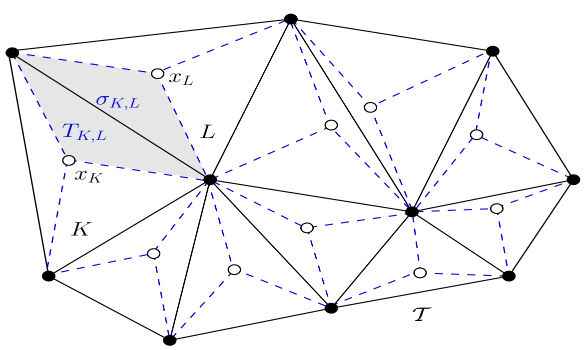

FIGURE 1.5 – Maillage Diamond

Ainsi,

$$\|\nabla_h u_h\|^2_{L^2(\Omega)} = \ell \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} |u_L - u_K|^2.$$

Nous supposerons qu'il existe une constante $a \in \mathbb{R}^+$, telle que pour tout élément K du maillage :

$$\min_{K\in\mathcal{T}, L\in N(K)} \frac{d_{K,L}}{\operatorname{diam}(K)} \geq a.$$

On note $\mathcal{D}$ une discrétisation admissible de Q_T, qui consiste à un maillage admissible de Ω, un pas de temps $\Delta t > 0$, et un nombre positif N choisit comme le plus petit entier tel que $N\Delta t \geq T$, et on note

$$t^n := n\Delta t \qquad \text{for } n \in \{0, \dots, N\}$$

La méthode des volumes finis consiste à intégrer le système de conservation de la masse (3.57)-(3.58) sur $]t_n, t_{n+1}[\times K$, on obtient les équations suivantes

$$\begin{aligned}&\int_{t_n}^{t_{n+1}} \int_K \phi \partial_t(\rho(p)s)\, dxdt - \int_{t_n}^{t_{n+1}} \int_{\partial K} \rho(p) M_1(s) \nabla p \cdot \eta_K \, d\sigma dt \\ &- \int_{t_n}^{t_{n+1}} \int_{\partial K} \rho(p) \nabla \beta(s) \cdot \eta_K \, d\sigma dt + \int_{t_n}^{t_{n+1}} \int_{\partial K} \rho^2(p) M_1(s) \mathbf{g} \cdot \eta_K \, d\sigma + \int_{t_n}^{t_{n+1}} \int_K \rho(p) s f_P \, dxdt = 0,\end{aligned} \tag{3.63}$$

$$\begin{aligned}&\int_{t_n}^{t_{n+1}} \int_K \phi \partial_t s \, dxdt + \int_{t_n}^{t_{n+1}} \int_{\partial K} M_2(s) \nabla p \cdot \eta_K \, d\sigma dt - \int_{t_n}^{t_{n+1}} \int_{\partial K} \nabla \beta(s) \cdot \eta_K \, d\sigma dt \\ &+ \int_{t_n}^{t_{n+1}} \int_{\partial K} \rho_2 M_2(s) \mathbf{g} \cdot \eta_K \, d\sigma dt + \int_{t_n}^{t_{n+1}} \int_K s f_P \, dxdt = \int_{t_n}^{t_{n+1}} \int_K (f_P - f_I) \, dxdt.\end{aligned} \tag{3.64}$$

où η_K est la normale unitaire à ∂K (frontière de K) dirigée vers l'extérieur de K. On note pour toute fonction $f(t,x)$ définie sur $(0,T) \times \Omega$ par f_K^n une approximation de $f(t_n, x_K)$.

Nous allons décrire brièvement une approximation de chaque terme des équations (3.57)-(3.58).

▷ Les conditions initiales :

$$p_0^K = \frac{1}{|K|} \int_K p_0(x)\, dx, \quad s_0^K = \frac{1}{|K|} \int_K s_0(x)\, dx$$

▷ Les termes d'évolutions :

$$\begin{aligned}\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_K \phi \partial_t s \; dxdt &= \frac{1}{\Delta t} \int_K \phi \Big(s(t_{n+1}, x) - s(t_n, x) \Big)\, dx \\ &\approx |K|\, \phi_K \frac{s_K^{n+1} - s_K^n}{\Delta t}\end{aligned}$$

De même,

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_K \phi\partial_t(\rho(p)s)\,dxdt = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_K \phi\Big(\rho(p(t_{n+1},x))s(t_{n+1},x)-\rho(p(t_n,x))s(t_n,x)\Big)\,dx$$
$$\approx |K|\,\phi_K\frac{\rho(p_K^{n+1})s_K^{n+1}-\rho(p_K^n)s_K^n}{\Delta t}$$

▷ Les termes capillaires. On considère un schéma implicite en temps et le maillage orthogonal admissible a été choisi pour donner une approximation simple pour les termes dissipatifs :

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\partial K}\nabla\beta(s)\cdot\eta_K\,d\sigma dt \approx \sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}(\beta(s_L^{n+1})-\beta(s_K^{n+1}))$$

De même,

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\partial K}\rho(p)\nabla\beta(s)\cdot\eta_K\,d\sigma dt \approx \sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\rho_{K,L}^{n+1}(\beta(s_L^{n+1})-\beta(s_K^{n+1}))$$

où

$$\rho_{K,L}^{n+1} = \begin{cases}\dfrac{1}{p_L^{n+1}-p_K^{n+1}}\displaystyle\int_{p_K^{n+1}}^{p_L^{n+1}}\rho(\xi)\,d\xi & \text{si } p_L^{n+1}-p_K^{n+1}\neq 0\\ \rho(p_K^{n+1}) & \text{sinon.}\end{cases}$$

Ce choix d'approximation de la densité aux interfaces joue un rôle essentiel pour contrôler le gradient discret de la pression globale.

▷ Les termes convectifs. Les termes de dissipation en pression sont vus comme des termes de convection des flux selon la vitesse "$-\nabla p$", ainsi un schéma amont est utilisé. On définit d'abord le gradient discret aux interfaces comme suit :

$$dp_{K,L} = \frac{|\sigma_{K,L}|}{d_{K,L}}\Big(p_L-p_K\Big) = (dp_{K,L})^+ - (dp_{K,L})^-$$

où $(dp_{K,L})^+ = \max(0, dp_{K,L})$ et $(dp_{K,L})^- = -\min(0, dp_{K,L})$.
La fonction M_2 est décroissante, le schéma amont s'écrit

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\partial K}M_2(s)\nabla p\cdot\eta_K\,d\sigma dt \approx \sum_{L\in N(K)}\int_{\sigma_{K,L}}M_2(s^{n+1})\nabla p^{n+1}\cdot\eta_{K,L}\,d\sigma,$$

et sur chaque interface

$$\int_{\sigma_{K,L}} M_2(s^{n+1})\nabla p^{n+1}\cdot\eta_{K,L} \approx \begin{cases} M_2(s_K^{n+1})dp_{K,L}^{n+1} & \text{si } dp_{K,L}^{n+1}\le 0 \\ M_2(s_L^{n+1})dp_{K,L}^{n+1} & \text{si } dp_{K,L}^{n+1} > 0 \end{cases}$$
$$= G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}),$$

avec

$$G_2(a,b,c) = M_2(b)c^+ - M_2(a)c^-. \tag{3.65}$$

De même, la fonction M_1 est croissante,

$$-\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\partial K}\rho(p)M_1(s)\nabla p\cdot\eta_K\,d\sigma dt \approx \sum_{L\in N(K)}\rho_{K,L}^{n+1}\Big(-M_1(s_L^{n+1})(dp_{K,L}^{n+1})^+ + M_1(s_K^{n+1})(dp_{K,L}^{n+1})^-\Big)$$
$$= \sum_{L\in N(K)}\rho_{K,L}^{n+1}\,G_1(s_K^{n+1}, s_L^{n+1}, dp_{K,L}^{n+1})$$

avec

$$G_1(a,b;q) = -M_1(b)c^+ + M_1(a)c^- \tag{3.66}$$

▷les termes de gravité :

$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\partial K}\rho^2(p)M_1(s)\mathbf{g}\cdot\eta_K\,d\sigma dt \approx \sum_{L\in N(K)}\Big(\rho^2(p_K^{n+1})M_1(s_K^{n+1})\mathbf{g}_{K,L} - \rho^2(p_L^{n+1})M_1(s_L^{n+1})\mathbf{g}_{L,K}\Big)$$
$$= \sum_{L\in N(K)} F_{1,K,L}^{n+1} = F_{1,K}^{n+1}$$
$$\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}\int_{\partial K}\rho_2 M_2(s)\mathbf{g}\cdot\eta_K\,d\sigma dt \approx \sum_{L\in N(K)}\Big(\rho_2 M_2(s_L^{n+1})\mathbf{g}_{K,L} - \rho_2 M_2(s_K^{n+1})\mathbf{g}_{L,K}\Big)$$
$$= \sum_{L\in N(K)} F_{2,K,L}^{n+1} = F_{2,K}^{n+1}$$

où $\mathbf{g}_{K,L} := \int_{K/L}(\mathbf{g}\cdot\eta_{K,L})^+\,d\sigma = \int_{K/L}(\mathbf{g}\cdot\eta_{L,K})^-\,d\gamma(x)$.
▷ Les termes sources :

$$f_{P,K}^{n+1} = \frac{1}{\Delta t|K|}\int_{t_n}^{t_{n+1}}\int_K f_P(t,x)\,dxdt, \quad f_{I,K}^{n+1} = \frac{1}{\Delta t|K|}\int_{t_n}^{t_{n+1}}\int_K f_I(t,x)\,dxdt.$$

On va maintenant décrire les propriétés des flux numériques G_i dans le cas général pour approcher les termes de convection. On peut remplacer le schéma amont par tout autre schéma à condition de satisfaire les propreiétés suivantes :

– **Monotonie**.

$$s \mapsto G_i(s,\cdot,\cdot) \text{ est croissante,} \tag{3.67}$$
$$s \mapsto G_i(\cdot,s,\cdot) \text{ est décroissante.} \tag{3.68}$$

cette propriété est importante pour assurer le principe du maximum sur les saturations.

– **Consistance**.

$$G_1(s,s,q) = -M_1(s)\,q, \qquad G_2(s,s,q) = M_2(s)\,q, \qquad \forall s,q. \tag{3.69}$$

– **Conservation**.

$$G_i(a,b,q) = -G_i(b,a,-q), \forall a,b,q \tag{3.70}$$

Cette condition assure la conservation des flux numériques et au point de vue mathématique permet l'intégration par parties, ce qui est essentiel pour la convergence du schéma numérique.

– **Couplage**. Il existe $m_0 > 0$ tel que

$$\Big(G_2(a,b,q) - G_1(a,b,q)\Big)q \geq m_0|q|^2, \qquad \text{for all } a,b,q \in \mathbb{R}. \tag{3.71}$$

Cette condition est spécifique à notre système, ceci traduit, au point de vue numérique, la version continue sur les flux. En effet, la version continue est celle donnée par la condition (3.69) :

$$\Big(G_2(s,s,q) - G_1(s,s,q)\Big)q = M(s)|q|^2 \geq m_0|q|^2, \qquad \text{for all } a,b,q \in \mathbb{R}.$$

Les flux numériques définies par (3.65),(3.66) vérifie (3.71), en effet

$$\Big(G_2(a,b,q) - G_1(a,b,q)\Big)q = M(b){q^+}^2 + M(a){q^-}^2 \geq m_0|q|^2.$$

Cette condition sur le couplage entre les flux assure que le gradient discret de la pression globale est borné.

En résumé, le schéma de volumes finis proposé pour la discrétisation du problème (3.57)-(3.58) consiste à chercher $P = (p_K^{n+1})_{K\in\mathcal{T},n\in[0,N]}$ et $S = (s_K^{n+1})_{K\in\mathcal{T},n\in[0,N]}$ solution de :

$$p_K^0 = \frac{1}{|K|}\int_K p_0(x)\,dx, \qquad s_K^0 = \frac{1}{|K|}\int_K s_0(x)\,dx, \tag{3.72}$$

et

$$|K|\,\phi_K \frac{\rho(p_K^{n+1})s_K^{n+1} - \rho(p_K^n)s_K^n}{\Delta t} - \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \rho_{K,L}^{n+1}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))$$
$$+ \sum_{L\in N(K)} \rho_{K,L}^{n+1} G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}) + F_{1,K}^{(n+1)} + |K|\,\rho(p_K^{n+1})s_K^{n+1} f_{P,K}^{n+1} = 0, \quad (3.73)$$

$$|K|\,\phi_K \frac{s_K^{n+1} - s_K^n}{\Delta t} - \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (\beta(s_L^{n+1}) - \beta(s_K^{n+1}))$$
$$+ \sum_{L\in N(K)} G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}) + F_{2,K}^{(n+1)} + |K|\,(s_K^{n+1} - 1) f_{P,K}^{n+1} = -\,|K|\, f_{I,K}^{n+1}. \quad (3.74)$$

Soit $(p_{\delta t,h},\ s_{\delta t,h}) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^2$ une solution discrète pour tout $K \in \mathcal{T}$ et $n \in [0, N]$

$$p_{\delta t,h}(t,x) = p_K^{n+1} \text{ et } s_{\delta t,h}(t,x) = s_K^{n+1}, \quad (3.75)$$

pour tout $x \in K$ et $t \in (n\Delta t, (n+1)\Delta t)$.

Le résultat principal de ce travail est le théorème suivant.
Theorem 1.3. *Sous les hypothèses (H1)-(H4). Soit $(p_0, s_0) \in L^2(\Omega,\mathbb{R})\times L^\infty(\Omega,\mathbb{R})$ et $0 \le s_0 \le 1$ p.p. dans Ω. Soit $(p_{\delta t,h}, s_{\delta t,h})$ une solution discrète du schéma VF (3.73)-(3.74). Alors, à une sous suite près, $(p_{\delta t,h}, s_{\delta t,h})$ converge vers une solution faible (p, s) comme $(\delta t, h) \to (0, 0)$ du problème (3.57)-(3.58) au sens de la Définition 1.1.*

La preuve de ce théorème de convergence se compose de plusieurs étapes :
Après l'existence de solutions par un théorème de point fixe et le principe du maximum sur la saturation, nous obtenons une estimation a priori sur le gradient discret de p et $\beta(s)$ comme suit

$$\sum_{K\in\mathcal{T}} |K|\, s_K^N \mathcal{H}(p_K^N) - \sum_{K\in\mathcal{T}} |K|\, s_K^0 \mathcal{H}(p_K^0)$$
$$+ \frac{1}{2}\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \left| p_K^{n+1} - p_L^{n+1} \right|^2 \le C \quad (3.76)$$

et

$$\begin{aligned}&\sum_{K\in\mathcal{T}}|K|\,B(s_K^N)-\sum_{K\in\mathcal{T}}|K|\,B(s_K^0)\\&+\frac{1}{4}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\left|\beta(s_K^{n+1})-\beta(s_L^{n+1})\right|^2\le C\end{aligned}\tag{3.77}$$

où $B'(s)=\beta(s)$, et $\mathcal{H}(p)=g(p)+\rho(p)p$ avec $g'(p)=-\rho(p)$.
Pour obtenir l'estimation (3.76), on multiplie l'équation discrète du gaz (3.73) et l'équation discrète de l'eau (3.74) respectivement par $p_K^{n+1} et\, g(p_K^{n+1})=\mathcal{H}(p_K^{n+1})-\rho(p_K^{n+1})p_K^{n+1}$; on additionne les équations et on somme sur K et n. Ensuite, pour obtenir l'estimation (3.77), on multiplie l'équation discrète de l'eau (3.74) par $\beta(s_K^{n+1})$ puis on somme sur K et n. De l'estimation (3.76) on déduit que

$$\nabla_h p_h \text{ est uniformément bornée dans } L^2(0,T,L^2(\Omega)).$$

et de (3.77)

$$\nabla_h\beta(s_h) \text{ est uniformément bornée dans } L^2(0,T,L^2(\Omega)).$$

Dans la deuxième étape, on montre des estimations sur les translatées en temps et en espace sur les suites s_h et $r_h=\rho(p_h)s_h$. On applique un théorème de compacité de type Riesz-Fréchet-Kolmogorov pour établir la convergence forte dans $L^1(Q_T)$.
L'aspect dégénéré en évolution sur la variable pression ne permet pas d'obtenir de compacité en variable pression. Par contre, le terme d'évolution $\partial_t(\rho(p)s)$ dans l'équation (3.57) nous permet d'obtenir de la compacité en variable $\rho(p)s$. D'autres difficultés techniques apparaissent alors, en particulier l'identification de la limite de la variable $\rho(p_h)s_h$ où (p_h,s_h) est solution du problème approché. Cette identification est rendue possible grâce à la monotonie de la fonction ρ, alors même que l'on ne dispose que de la convergence faible dans la variable pression et de la convergence forte sur la saturation dans L^2.
Enfin, le passage à la limite est rendu possible grâce à la convergence forte de $\rho(p_h)\beta(s_h)$ vers $\rho(p)\beta(s)$ dans $L^q(Q_T)$, pour tout $1\le q<\infty$.

CHAPITRE 2

On a fully nonlinear degenerate parabolic system modeling immiscible gas-water displacement in porous media

Abstract. In this paper, we are interested in the simultaneous flow of two immiscible fluid phases within a porous medium. We consider two-phase flow model where the fluids are immiscible and there is no mass transfer between the phases. The medium is saturated by compressible/incompressible phase flows. We study the gas-water displacement without simplified assumptions on state law of gas density. We establish an existence result for the nonlinear degenerate parabolic system based on new energy estimate on pressures.

1 Introduction, Assumptions and Main results

The equations describing the immiscible displacement of two compressible fluids are given by the following mass conservation of each phase :

$$\phi(x)\partial_t(\rho_i(p_i)s_i) + \operatorname{div}(\rho_i(p_i)\mathbf{V}_i) + \rho_i(p_i)s_i f_P(t,x) = \rho_i(p_i)s_i^I f_I(t,x) \tag{1.1}$$

where ϕ is the porosity of the medium, ρ_i and s_i are respectively the density and the saturation of the i^{th} fluid. Here the functions f_I and f_P are respectively the injection and production terms. Note that in equation (1.1) the injection term is multiplied by a known saturation s_i^I corresponding to the known injected fluid, whereas the production term is multiplied by the unknown saturation s_i corresponding to the produced fluid.

We are concerned with the study of (1.1). considering one injected fluid (i.e $s_1^I =$

0, $s_2^I = 1$), and one incompressible fluid (i.e $\rho_2(p_2) = \rho_2 \in \mathbb{R}^+$),

$$\phi(x)\partial_t(\rho_1(p_1)s_1)(t,x) + \mathrm{div}(\rho_1(p_1)\mathbf{V}_1)(t,x) + \rho_1(p_1)s_1 f_P(t,x) = 0, \tag{1.2}$$

$$\phi(x)\partial_t s_2(t,x) + \mathrm{div}(\mathbf{V}_2)(t,x) + s_2 f_P(t,x) = f_I(t,x). \tag{1.3}$$

Theoretical analysis has been studied by many authors for miscible/immiscible and incompressible/compressible flows in porous media. The study of the miscible flow models has been investigated in ([10], [11], [40]) and recently in ([17], [18], [19]). The immiscible and incompressible flows have been treated by many authors ([10], [9], [22], [39], [29], [41], [42]). For two immiscible compressible flows, we refer to [44], [47], and recently [45] and [15].
The velocity of each fluid $\mathbf{V}_i$ is given by the Darcy law :

$$\mathbf{V}_i = -\mathbf{K}\frac{k_i(s_i)}{\mu_i}\Big(\nabla p_i - \rho_i(p_i)\mathbf{g}\Big), \qquad i = 1, 2. \tag{1.4}$$

where $\mathbf{K}(x)$ is the permeability tensor of the porous medium at point x to the fluid under consideration, $k_i(s_i)$ the relative permeability of the i^{th} phase, μ_i the constant i-phase's viscosity and p_i the i-phase's pressure and $\mathbf{g}$ is the gravity term. By definition of saturations, one has

$$s_1(t,x) + s_2(t,x) = 1. \tag{1.5}$$

The curvature of the contact surface between the two fluids links the jump of pressure of the two phases to the saturation by the capillary pressure law in order to close the system (1.1)-(1.5),

$$f(s_1) = p_1 - p_2. \tag{1.6}$$

With the arbitrary choice of (1.6) (the jump of pressure is a function of s_1), the application $s_1 \mapsto f(s_1)$ is non-decreasing, ($\frac{df}{ds_1}(s_1) > 0$, for all $s_1 \in [0,1]$), and usually $f = 0$, in the case of two incompressible phases or two-phase compressible incompressible, when the wetting fluid is at its maximum saturation. In order to know which of the fluids is the wetting one, one has to look at the meniscus separating the two fluids in a capillary tube, the concavity of the meniscus is oriented towards the non wetting fluid. For example, air is the non wetting phase in water air displacement. In this study we consider the index $i = 1$ represents the non-wetting fluid. With the choice (1.6), f will always be an increasing function of s_1 defined over the interval $[0;1]$, and vanishing when $s_1 = 0$.

In section 4 we will use the feature of global pressure. For that let us denote,

$$
\begin{aligned}
&M_i(s_i) = k_i(s_i)/\mu_i && i - \text{phase's mobility,}\\
&M(s_1) = M_1(s_1) + M_2(1 - s_1) && \text{the total mobility}
\end{aligned}
$$

and as in [22], [64] and [45] we define the functions $\tilde{p}(s_1)$, $\bar{p}(s_1)$ such that

$$\tilde{p}'(s_1) = \frac{M_1(s_1)}{M(s_1)} f'(s_1), \qquad \bar{p}'(s_1) = -\frac{M_2(s_2)}{M(s_1)} f'(s_1), \tag{1.7}$$

so, the global pressure is defined as $p = p_2 + \tilde{p}(s_1)$ or equivalently $p = p_1 + \bar{p}(s_1)$. Finally, let us denote the capillary term by

$$\alpha(s_1) = \frac{M_1(s_1) M_2(s_2)}{M(s_1)} f'(s_1) \geq 0,$$

and define

$$\beta(s) = \int_0^s \alpha(\xi) d\xi. \tag{1.8}$$

The study of two immiscible compressible models has done in [44, 45, 46, 47]. The authors consider a formulation in phase pressure and saturation and restrict the dependence of the densities on global pressure. In this paper, we consider gas-water model (1.2)-(1.3) under the formulation in phase pressures, this formulation was employed in the simultaneous solution scheme in petroleum reservoirs (Douglas, Peaceman, and Rachford, 1959)[28]. The model is treated without simplified assumptions on the gas density, we consider that the gas density depends on its corresponding pressure. We derive new energy estimates on the velocities. Nevertheless, these estimates are degenerated in the sense that they don't permit the control of gradients of pressure of each phase, especially when a phase is not locally present in the domain. So, the global pressure has a major role in the analysis, we will show that the control of the velocities ensures the control of the global pressure in the whole domain regardless of the presence or the disappearance of the phases.

We detail the physical context by introducing the boundary conditions, the initial conditions and some assumptions on the data of the problem.
Let $T > 0$, fixed and let Ω be a bounded set of $\mathbb{R}^d$ $(d \geq 1)$. We set $Q_T = (0, T) \times \Omega$, $\Sigma_T = (0, T) \times \partial\Omega$. To the system (1.1)-(1.5)-(1.6) $(i = 1, 2)$, we add the following mixed boundary conditions and initial conditions. We consider the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_{imp}$, where Γ_1 denotes the injection boundary of the first

phase and Γ_{imp} the impervious one.

$$\begin{cases} p_1(t,x) = 0, \;\; p_2(t,x) = 0 & \text{on } \Gamma_1 \\ \mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0 & \text{on } \Gamma_{imp} \end{cases} \tag{1.9}$$

where $\mathbf{n}$ is the outward normal to the boundary Γ_{imp}.
The initial conditions are defined on pressures

$$\begin{cases} p_1(0,x) = p_1^0(x) \text{ in } \Omega \\ p_2(0,x) = p_2^0(x) \text{ in } \Omega. \end{cases} \tag{1.10}$$

Next we are going to introduce some physically relevant assumptions on the coefficients of the system.

($H1$) The porosity $\phi \in W^{1,\infty}(\Omega)$ and there is two positive constants ϕ_0 and ϕ_1 such that $\phi_0 \leq \phi(x) \leq \phi_1$ almost everywhere $x \in \Omega$.

($H2$) There exist two positive constants k_0 and k_∞ such that for all $\xi \in \mathbb{R}^d$, almost everywhere $x \in \Omega$.

$$\|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \leq k_\infty \quad \text{and} \quad (\mathbf{K}(x)\xi, \xi) \geq k_0|\xi|^2$$

($H3$) The functions M_1 and M_2 belong to $\mathcal{C}^0([0,1];\mathbb{R}^+)$, $M_1(s_1 = 0) = 0$ and $M_2(s_2 = 0) = 0$. In addition, there is a strictly positive constant m_0, such that, for all $s_1 \in [0,1]$,

$$M_1(s_1) + M_2(s_2) \geq m_0.$$

($H4$) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \geq 0$ almost everywhere $(t,x) \in Q_T$.

($H5$) The density ρ_1 is $\mathcal{C}^2(\mathbb{R})$, increasing and there exist two positive constants ρ_m and ρ_M such that $0 < \rho_m \leq \rho_1(p_1) \leq \rho_M$.

($H6$) The capillary pressure function $f \in \mathcal{C}^0([0,1];\mathbb{R}^+)$, monotone increasing, f is differentiable on $[0,1[$ and $0 < \underline{f} \leq \frac{df}{ds}$.

($H7$) The function $\alpha \in \mathcal{C}^2([0,1];\mathbb{R}^+)$ satisfies $\alpha(s) > 0$ for $0 < s < 1$, and $\alpha(0) = 0$.
We assume that β^{-1}, inverses of $\beta(s) := \int_0^s \alpha(z)dz$, is Hölder function of order θ, with $0 < \theta \leq 1$, on $[0, \beta(1)]$.

The assumptions (H1)–(H7) are classical for porous media. The assumption (H7) on β^{-1} indicates that the mobilities are polynomial functions around $s_1 = 0$. According to the definition of α and (H3), we have

$$\alpha(s_1 = 1) = M_2(s_1 = 1)f'(s_1 = 1) > 0, \tag{1.11}$$

which shows that the behavior of $f'(s_1)$ is equivalent to $\frac{1}{M_2(s_1)}$ around $s_1 = 1$, then $f'(s_1) \to \infty$ when $s_1 \to 1$. Note that, due to the boundedness of the capillary pressure function, the functions $\tilde{p}$ and $\bar{p}$ (defined in (1.7)) are bounded on $[0\,;\,1]$.

The main existence result of this paper is given below, for that let us define the following Sobolev space

$$H^1_{\Gamma_1}(\Omega) = \{\, u \in H^1(\Omega)\,;\, u = 0 \text{ on } \Gamma_1 \,\},$$

this is an Hilbert space when equipped with the norm $\|u\|_{H^1_{\Gamma_1}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.
Let us state the main results of this paper.

Theorem 2.1. *Let (H1)-(H7) hold. Let $(p_1^0,\, p_2^0)$ belongs to $L^2(\Omega) \times L^2(\Omega)$. Then there exists (p_1, p_2) satisfying*

$$s_1 \in L^{2/\theta}(0,T; W^{\tau\theta,2/\theta}(\Omega)) \text{ for some } 0 < \tau < 1;\ s_1 = 0 \text{ on } \Gamma_1 \tag{1.12}$$

$$p_i \in L^2(Q_T), \sqrt{M_i(s_i)}\nabla p_i \in (L^2(Q_T))^d \tag{1.13}$$

$$\phi\partial_t(\rho_1(p_1)s_1) \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))'),\ \phi\partial_t s_1 \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))') \tag{1.14}$$

$$0 \le s_i(t,x) \le 1 \text{ a.e in } Q_T,\ \ i = 1,2,\ \ \beta(s_1) \in L^2(0,T;H^1_{\Gamma_1}(\Omega)) \tag{1.15}$$

such that for all $\varphi,\, \xi \in C^1(0,T;H^1_{\Gamma_1}(\Omega))$ with $\varphi(T) = \xi(T) = 0$,

$$\begin{aligned} &-\int_{Q_T} \phi\rho_1(p_1)s_1\partial_t\varphi\,dxdt - \int_\Omega \phi(x)\rho_1(p_1^0(x))s_1^0(x)\varphi(0,x)\,dx \\ &+\int_{Q_T} \mathbf{K}M_1(s_1)\rho_1(p_1)\nabla p_1\cdot\nabla\varphi\,dxdt - \int_{Q_T} \mathbf{K}M_1(s_1)\rho_1^2(p_1)\mathbf{g}\cdot\nabla\varphi\,dxdt \\ &+\int_{Q_T} \rho_1(p_1)s_1 f_P\varphi\,dxdt = 0, \end{aligned} \tag{1.16}$$

$$\begin{aligned} &-\int_{Q_T} \phi s_2\partial_t\xi\,dxdt - \int_\Omega \phi(x)s_2^0(x)\xi(0,x)\,dx \\ &+\int_{Q_T} \mathbf{K}M_2(s_2)\nabla p_2\cdot\nabla\xi\,dxdt - \int_{Q_T} \mathbf{K}M_2(s_2)\rho_2\mathbf{g}\cdot\nabla\xi\,dxdt \\ &+\int_{Q_T} s_2 f_P\xi\,dxdt = \int_{Q_T} f_I\,\xi\,dxdt, \end{aligned} \tag{1.17}$$

and finally the initial conditions are satisfied in a weak sense as follows :

$$\text{For all } \psi \in H^1_{\Gamma_1}(\Omega) \text{ the functions } t \longrightarrow \int_\Omega \phi\rho_1(p_1)s_1\psi\,dx \in \mathcal{C}^0([0,T]), \tag{1.18}$$

$$\text{and } t \longrightarrow \int_\Omega \phi s_2\psi\,dx \in \mathcal{C}^0([0,T]) \tag{1.19}$$

furthermore we have

$$\Big(\int_\Omega \phi\rho_1(p_1)s_1\psi\,dx\Big)(0) = \int_\Omega \phi\rho_1(p_1^0)s_1^0\psi\,dx \tag{1.20}$$

$$\Big(\int_\Omega \phi s_2\psi\,dx\Big)(0) = \int_\Omega \phi s_2^0\psi\,dx. \tag{1.21}$$

The notion of weak solutions is very natural provided that we explain the origin of the requirements (1.14)–(1.15) Obviously, they correspond to *a priori* estimates. Indeed, (1.16)-(1.17) ensure that $s_i \geq 0$ $(i = 1, 2)$ which is equivalent to $0 \leq s_i \leq 1$. (the proof is detailed in lemma 2.5). The key point is to obtain the estimates on ∇p and $\nabla\beta(s_1)$.
For that, define

$$g_1(p_1) := \int_0^{p_1} \frac{1}{\rho_1(\xi)}\,d\xi, \tag{1.22}$$

$$\mathcal{H}_1(p_1) := \rho_1(p_1)g_1(p_1) - p_1, \tag{1.23}$$

then $\mathcal{H}_1'(p_1) = \rho_1'(p_1)g_1(p_1),\ \mathcal{H}_1(0) = 0,\ \mathcal{H}_1(p_1) \geq 0$ for all p_1, and $\mathcal{H}_1$ is sublinear with respect to p_1 (i.e $|\mathcal{H}_1(p_1)| \leq C|p_1|$).
Multiplying (1.2) by $g_1(p_1)$ and (1.3) by p_2 then integrate the equations with respect to x and adding them, we deduce at least formally,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi\left(s_1\mathcal{H}_1(p_1) + \int_0^{s_1} f(\xi)\,d\xi\right)dx + \int_\Omega \mathbf{K}M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx \\
&+\int_\Omega \mathbf{K}M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx - \int_\Omega \mathbf{K}M_1(s_1)\rho_1(p_1)\mathbf{g}\cdot\nabla p_1\,dx - \int_\Omega \mathbf{K}M_2(s_2)\rho_2\mathbf{g}\cdot\nabla p_2\,dx \\
&\qquad + \int_\Omega \rho_1(p_1)s_1 f_P g_1(p_1)\,dx + \int_\Omega s_2 f_P p_2\,dx = \int_\Omega f_I p_2\,dx.
\end{aligned}$$

Using the assumptions (H1)–(H6) and the fact that $\mathcal{H}_1 \geq 0$, $g_1(p_1)$ is sublinear with respect to p_1 we deduce

$$\int_{Q_T} M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx + \int_{Q_T} M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx < \infty. \tag{1.24}$$

We have

$$\nabla p = \nabla p_2 + \frac{M_1}{M}\nabla f(s_1) = \nabla p_1 - \frac{M_2}{M}\nabla f(s_1), \tag{1.25}$$

and consequently, we have the main equality

$$\begin{aligned}
&\int_{Q_T} M|\nabla p|^2\,dx + \int_{Q_T}\frac{M_1M_2}{M}|\nabla f(s_1)|^2\,dx = \\
&\qquad\int_{Q_T} M_1(s_1)\nabla p_1\cdot\nabla p_1\,dx + \int_{Q_T} M_2(s_2)\nabla p_2\cdot\nabla p_2\,dx.
\end{aligned} \tag{1.26}$$

Due to the fact that the total mobility does not vanish, the above equality permits

the control of global pressure in the whole domain whereas the control on the gradient of pressure of each phase is not available in the region where the phase is not presented. Also, the second integral in the left hand side gives a control on a function of capillary term. We will see in section 4, the control of the global pressure and the capillary terms are sufficient to give a sense of each pressure almost everywhere in the domain.
Before establishing theorem 2.1, we introduce the existence of regularized solutions to system (1.1). Firstly we are interested on non degenerate system by adding a dissipative term on saturation preserving a maximum principle on saturations. Precisely, we consider the non-degenerate system :

$$\begin{aligned}\phi\partial_t(\rho_1(p_1^\eta)s_1^\eta) - \operatorname{div}(\mathbf{K}\rho_1(p_1^\eta)M_1(s_1^\eta)\nabla p_1^\eta) + \operatorname{div}(\mathbf{K}\rho_1^2(p_1^\eta)M_1(s_1^\eta)\mathbf{g})& \\ - \eta\operatorname{div}(\rho_1(p_1^\eta)\nabla(p_1^\eta - p_2^\eta)) + \rho_1(p_1^\eta)s_1^\eta f_P = 0,& \quad (1.27)\end{aligned}$$

$$\begin{aligned}\phi\partial_t(s_2^\eta) - \operatorname{div}(\mathbf{K}M_2(s_2^\eta)\nabla p_2^\eta) + \operatorname{div}(\mathbf{K}\rho_2 M_2(s_2^\eta)\mathbf{g})& \\ - \eta\operatorname{div}(\nabla(p_2^\eta - p_1^\eta)) + s_2^\eta f_P = f_I,& \quad (1.28)\end{aligned}$$

completed with the initial conditions (1.10), and the following mixed boundary conditions,

$$\begin{cases} p_1^\eta(t,x) = 0, \;\; p_2^\eta(t,x) = 0 & \text{on } (0,T)\times\Gamma_1 \\ \Big(\mathbf{K}M_1(s_1^\eta)(\nabla p_1^\eta - \rho_1(p_1^\eta)\mathbf{g}) + \eta\nabla(p_1^\eta - p_2^\eta)\Big)\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{imp} \\ \Big(\mathbf{K}M_2(s_2^\eta)(\nabla p_2^\eta - \rho_2\mathbf{g}) - \eta\nabla(p_1^\eta - p_2^\eta)\Big)\cdot\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{imp} \end{cases} \quad (1.29)$$

where $\mathbf{n}$ is the outward normal to the boundary Γ_{imp}.

Existence of solutions of the above system is given in the following theorem.
Theorem 2.2. *Let (H1)-(H6) hold. Let $(p_1^0,\, p_2^0)$ belongs to $L^2(\Omega)\times L^2(\Omega)$. Then for all $\eta > 0$, there exists (p_1^η, p_2^η) satisfying*

$$\begin{aligned}& p_i^\eta \in L^2(0,T;H^1_{\Gamma_1}(\Omega)),\\ & \phi\partial_t(\rho_1(p_1^\eta)s_1^\eta) \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))'), \phi\partial_t s_2^\eta \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))'),\\ & \rho_1(p_1^\eta)s_1^\eta \in C^0([0,T];L^2(\Omega)), s_1^\eta \in L^2(0,T;H^1_{\Gamma_1}(\Omega)),\\ & s_2^\eta \in \Big(L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))\Big),\\ & 0 \le s_i^\eta(t,x) \le 1 \;\; a.e \; in \; Q_T, \;\; i = 1,2,\end{aligned}$$

for all $\varphi, \xi \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$,

$$\begin{aligned}&\langle \phi\partial_t(\rho_1(p_1^\eta)s_1^\eta),\varphi\rangle + \int_{Q_T} \mathbf{K}M_1(s_1^\eta)\rho_1(p_1^\eta)\nabla p_1^\eta\cdot\nabla\varphi\,dxdt\\ &-\int_{Q_T}\mathbf{K}M_1(s_1^\eta)\rho_1^2(p_1^\eta)\mathbf{g}\cdot\nabla\varphi\,dxdt + \eta\int_{Q_T}\rho_1(p_1^\eta)\nabla(p_1^\eta-p_2^\eta)\cdot\nabla\varphi\,dxdt\\ &+\int_{Q_T}\rho_1(p_1^\eta)s_1^\eta f_P\varphi\,dxdt = 0,\end{aligned} \tag{1.30}$$

$$\begin{aligned}&\langle \phi\partial_t(s_2^\eta),\xi\rangle + \int_{Q_T}\mathbf{K}M_2(s_2^\eta)\nabla p_2^\eta\cdot\nabla\xi\,dxdt - \int_{Q_T}\mathbf{K}M_2(s_2^\eta)\rho_2\mathbf{g}\cdot\nabla\xi\,dxdt\\ &-\eta\int_{Q_T}\nabla(p_1^\eta-p_2^\eta)\cdot\nabla\xi\,dxdt + \int_{Q_T}s_2^\eta f_P\xi\,dxdt = \int_{Q_T} f_I\xi\,dxdt\end{aligned} \tag{1.31}$$

where the bracket $\langle\cdot,\cdot\rangle$ *represents the duality product between* $L^2(0,T;(H^1_{\Gamma_1}(\Omega))')$ *and* $L^2(0,T;H^1_{\Gamma_1}(\Omega))$. *Furthermore, the initial conditions are satisfied :*

$$(\rho_1(p_1^\eta)s_1^\eta)(0,x) = \rho_1(p_1^0)s_1^0(x) \text{ and } s_2^\eta(0,x) = s_2^0(x) \text{ a.e. in } \Omega.$$

The proof of the theorem 2.2 is based on time discretization method. The construction of this method is described in section 3. The main idea of this method consists firstly to solve elliptic system on each interval of discretization. Then, we reconstruct a solution for the parabolic system. Now, we introduce the existence of solutions to a time discretization of (1.27)-(1.28),

$$\begin{aligned}&\phi\frac{\rho_1(p_1)s_1-\rho_1^\star s_1^\star}{h} - \operatorname{div}(\mathbf{K}\rho_1(p_1)M_1(s_1)\nabla p_1) + \operatorname{div}(\mathbf{K}M_1(s_1)\rho_1^2\mathbf{g})\\ &\qquad - \eta\operatorname{div}(\rho_1(p_1)\nabla(p_1-p_2)) + \rho_1(p_1)s_1f_P = 0,\end{aligned} \tag{1.32}$$

$$\begin{aligned}&\phi\frac{s_2-s_2^\star}{h} - \operatorname{div}(\mathbf{K}M_2(s_2)\nabla p_2) + \operatorname{div}(\mathbf{K}M_2(s_2)\rho_2\mathbf{g})\\ &\qquad - \eta\operatorname{div}(\nabla(p_2-p_1)) + s_2f_P = f_I,\end{aligned} \tag{1.33}$$

where $\rho_1^\star$ and $s_i^\star$, formally, are the values of the $h-$translated in time of $\rho_1(p_1)$ and s_i respectively, $i=1,2$.

Existence of solutions of system (2.1)(2.2) is given in the following theorem.

Theorem 2.3. *Let (H1)-(H6) hold. Let* $0\le\rho_1^\star(x)\le\rho_M$, $0\le s_1^\star(x)\le 1$ *be defined almost everywhere in* Ω. *Then for all* $h>0$, *there exists* $(p_1^h,p_2^h) = (p_1^{\eta,h},p_2^{\eta,h})$ *satisfying*

$$\begin{aligned}&p_1^h\in H^1_{\Gamma_1}(\Omega),\ p_2^h\in H^1_{\Gamma_1}(\Omega),\\ &s_1^h\in H^1(\Omega),\ s_2^h\in H^1_{\Gamma_1}(\Omega),\ 0\le s_i^h(t,x)\le 1 \text{ a.e in } Q_T,\ i=1,2,\end{aligned}$$

for all $\varphi, \xi \in H^1_{\Gamma_1}(\Omega)$,

$$\begin{aligned}&\int_\Omega \phi \frac{\rho_1(p_1^h)s_1^h - \rho_1^\star s_1^\star}{h}\varphi\, dx + \int_\Omega \mathbf{K} M_1(s_1^h)\rho_1(p_1^h)\nabla p_1^h \cdot \nabla\varphi\, dx\\ &- \int_\Omega \mathbf{K} M_1(s_1^h)\rho_1^2(p_1^h)\mathbf{g}\cdot\nabla\varphi\, dx + \eta\int_\Omega \rho_1(p_1^h)\nabla(p_1^h - p_2^h)\cdot\nabla\varphi\, dx\\ &+ \int_\Omega \rho_1(p_1^h)s_1^h f_P\varphi\, dx = 0,\end{aligned} \tag{1.34}$$

$$\begin{aligned}&\int_\Omega \phi\frac{s_2^h - s_2^\star}{h}\xi\, dx + \int_\Omega \mathbf{K} M_2(s_2^h)\nabla p_2^h\cdot\nabla\xi\, dx - \int_\Omega \mathbf{K} M_2(s_2^h)\rho_2\mathbf{g}\cdot\nabla\xi\, dx\\ &- \eta\int_\Omega \nabla(p_1^h - p_2^h)\cdot\nabla\xi\, dx + \int_\Omega s_2^h f_P\xi\, dx = \int_\Omega f_I\xi\, dx,\end{aligned} \tag{1.35}$$

The rest of the paper is organized as follows. In the next section we deal with the time discrete model (2.1)-(2.2) in two steps. The first step deals with an elliptic system with non degenerate mobilities, $M_i^\varepsilon = M_i + \varepsilon$ with $\varepsilon > 0$, in this step we apply a suitable fixed point theorem, Leray-Schauder, to get weak solution. The second step is to pass to the limit as ε goes to zero depending on a suitable uniform estimate (w. r. to ε), and a maximum principle ensures the positivity of saturations which achieves the proof of theorem 2.3.
In the third section we introduce a sequence of solutions solving (1.34) (1.35). This choice is motivated by the fact that no evolution have to be considered in a first step. The problem of degeneracy of evolution term is temporarily sat aside. Furthermore, the maximum principle is conserved on saturation after the passage to the limit on in the non linear variational elliptic system. The last section is devoted to pass from non-degenerate case to degenerate case. through a compactness *lemma* which allow us with the help of some estimates to pass the limit and end the proof of existence of weak solutions of the system under consideration.
The next section is devoted to the analysis of the elliptic problem.

2 Study of a nonlinear elliptic system (proof of theorem 2.3)

Having in mind a time discretization of (1.27)-(1.28), we are concerned with the following system,

$$\begin{aligned}\phi\frac{\rho_1(p_1)s_1 - \rho_1^\star s_1^\star}{h} - \operatorname{div}(\mathbf{K}\rho_1(p_1)M_1(s_1)\nabla p_1) + \operatorname{div}(\mathbf{K}M_1(s_1)\rho_1^2\mathbf{g})&\\ - \eta\operatorname{div}(\rho_1(p_1)\nabla(p_1 - p_2)) + \rho_1(p_1)s_1 f_P = 0,&\end{aligned} \tag{2.1}$$

$$\phi\frac{s_2 - s_2^\star}{h} - \operatorname{div}(\mathbf{K}M_2(s_2)\nabla p_2) + \operatorname{div}(\mathbf{K}M_2(s_2)\rho_2\mathbf{g}) \\ - \eta \operatorname{div}\nabla(p_2 - p_1) + s_2 f_P = f_I, \quad (2.2)$$

where $\rho_1^\star$ and $s_i^\star$, formally, are the values of the $h-$translated in time of $\rho_1(p_1)$ and s_i respectively, $i = 1, 2$.

In order to show the existence of solutions of the system (2.1)-(2.2), we introduce two regularisations. The first consists to replace the mobilities M_i, $(i = 1, 2.)$, by a non-degenerate positive mobilities

$$M_i^\varepsilon = M_i + \varepsilon, \quad i = 1, 2, \text{ and } \varepsilon > 0.$$

The second consists to trunk high frequencies of nonlinear elliptic term in pressures. For that, let $\mathcal{P}_N$ be the orthogonal projector of $L^2(\Omega)$ on the first N eigenfunctions $\{p_1, \cdots, p_N\}$ of the eigenproblem

$$\begin{cases} -\Delta p_i = \lambda_i p_i & \text{in } \Omega \\ p_i = 0 & \text{on } \Gamma_1 \\ \nabla p_i \cdot \mathbf{n} = 0 & \text{on } \Gamma_{imp} \end{cases} \quad (2.3)$$

The projector $\mathcal{P}_N$ appears in (2.9) to make regular the implied term. The necessity of this regularization appears in the coming proposition in order to define the operator which we apply on the Leray-Schauder fixed point theorem.

The addition of such ε to the mobilities lead to the loss of maximum principle on the saturations s_i $(i = 1, 2.)$ so the functions M_1 and M_2 are extended on $\mathbb{R}$ by continuous constant functions outside $[0, 1]$ and then are bounded on $\mathbb{R}$. For the same reason we denote,

$$Z(s) = \begin{cases} 0 & \text{for } s \leq 0 \\ s & \text{for } s \in [0, 1] \\ 1 & \text{for } s \geq 1. \end{cases} \quad (2.4)$$

In the same spirit and in order to write the saturations s_i $(i = 1, 2.)$ as functions of the principle unknowns p_1 and p_2 of the system, we extend the capillary pressure function f into $\bar{f}$ where the function

$$\bar{f} \text{ continuous, bounded and strict monotony outside } [0, 1], \quad (2.5)$$

this is possible in the case when the capillary function f is bounded, in other words when $\mid f(0) \mid < \infty$, and denote by $s_1 = \bar{f}^{-1}(p_1 - p_2)$ and $s_2 = 1 - \bar{f}^{-1}(p_1 - p_2)$.

Note that, the extended functions can be written

$$M_1(s_1) := M_1(Z(s_1)), \quad M_2(s_2) := M_2(Z(s_2)), \tag{2.6}$$
$$\alpha(s_1) := \alpha(Z(s_1)), \tag{2.7}$$

$$\beta(s_1) := \int_0^{s_1} \alpha(Z(s))\,ds = \begin{cases} 0 & \text{if } s_1 \le 0 \\ \int_0^{s_1} \alpha(s)\,ds & \text{if } 0 \le s_1 \le 1 \\ \beta(1) + \alpha(1)(s_1 - 1) & \text{if } s_1 \ge 1. \end{cases} \tag{2.8}$$

Existence of solution to (2.1)-(2.2) is constructed in three steps. The first one consists in studying the following problem for fixed parameters $\varepsilon > 0$ and $N > 0$,

$$\int_\Omega \phi \frac{\rho_1(p_1^{\varepsilon,N}) Z(s_1^{\varepsilon,N}) - \rho_1^\star s_1^\star}{h} \varphi\,dx + \int_\Omega \mathbf{K} M_1^\varepsilon(s_1^{\varepsilon,N}) \rho_1(p_1^{\varepsilon,N}) \nabla p_1^{\varepsilon,N} \cdot \nabla\varphi\,dx$$
$$- \int_\Omega \mathbf{K} M_1(s_1^{\varepsilon,N}) \rho_1^2(p_1^{\varepsilon,N}) \mathbf{g} \cdot \nabla\varphi\,dx + \eta \int_\Omega \rho_1(p_1^{\varepsilon,N}) \nabla(\mathcal{P}_N p_1^{\varepsilon,N} - \mathcal{P}_N p_2^{\varepsilon,N}) \cdot \nabla\varphi\,dx$$
$$+ \int_\Omega \rho_1(p_1^{\varepsilon,N}) Z(s_1^{\varepsilon,N}) f_P \varphi\,dx = 0, \tag{2.9}$$

$$\int_\Omega \phi \frac{Z(s_2^{\varepsilon,N}) - s_2^\star}{h} \xi\,dx + \int_\Omega \mathbf{K} M_2^\varepsilon(s_2^{\varepsilon,N}) \nabla p_2^{\varepsilon,N} \cdot \nabla\xi\,dx$$
$$- \int_\Omega \mathbf{K} M_2(s_2^{\varepsilon,N}) \rho_2 \mathbf{g} \cdot \nabla\xi\,dx - \eta \int_\Omega \nabla(\mathcal{P}_N p_1^{\varepsilon,N} - \mathcal{P}_N p_2^{\varepsilon,N}) \cdot \nabla\xi\,dx$$
$$+ \int_\Omega Z(s_2^{\varepsilon,N}) f_P \xi\,dx = \int_\Omega f_I \xi\,dx, \tag{2.10}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.
The second step concerns the passage to the limit as N goes to infinity in order to recover the full physical diffusion on pressures p_1 and p_2, while the third one is the passage to the limit as ε goes to zero.

Step 1. We show for fixed $N > 0$ and $\varepsilon > 0$ existence of solutions to (2.9)-(2.10). We omit for the time being the dependence of solutions on parameter $N > 0$ and ε.

Proposition 2.1. *Assume $\rho_i^\star s_i^\star$ belongs to $L^2(\Omega)$ and $\rho_i^\star s_i^\star \ge 0$, Then there exists (p_1, p_2) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, solution of (2.9)-(2.10).*

Démonstration. The proof is based on the Leray-Schauder fixed point theorem.

Let $\mathcal{T}$ be a map from $L^2(\Omega) \times L^2(\Omega)$ to $L^2(\Omega) \times L^2(\Omega)$ defined by

$$\mathcal{T}(\overline{p}_1, \overline{p}_2) = (p_1, p_2),$$

where the pair (p_1, p_2) is the unique solution of the system (2.11)-(2.12)

$$\begin{aligned} \int_\Omega \phi \frac{\rho_1(\overline{p}_1) Z(\overline{s}_1) - \rho_1^\star s_1^\star}{h} \varphi\, dx + \int_\Omega \mathbf{K} M_1^\varepsilon(\overline{s}_1) \rho_1(\overline{p}_1) \nabla p_1 \cdot \nabla \varphi\, dx \\ - \int_\Omega \mathbf{K} M_1(\overline{s}_1) \rho_1^2(\overline{p}_1) \mathbf{g} \cdot \nabla \varphi\, dx + \eta \int_\Omega \rho_1(\overline{p}_1) \nabla(\mathcal{P}_N \overline{p}_1 - \mathcal{P}_N \overline{p}_2) \cdot \nabla \varphi\, dx \\ + \int_\Omega \rho_1(\overline{p}_1) Z(\overline{s}_1) f_P \varphi\, dx = 0, \quad (2.11) \end{aligned}$$

$$\begin{aligned} \int_\Omega \phi \frac{Z(\overline{s}_2) - s_2^\star}{h} \xi\, dx + \int_\Omega \mathbf{K} M_2^\varepsilon(\overline{s}_2) \nabla p_2 \cdot \nabla \xi\, dx - \int_\Omega \mathbf{K} M_2(\overline{s}_2) \rho_2 \mathbf{g} \cdot \nabla \xi\, dx \\ - \eta \int_\Omega \nabla(\mathcal{P}_N \overline{p}_1 - \mathcal{P}_N \overline{p}_2) \cdot \nabla \xi\, dx + \int_\Omega Z(\overline{s}_2) f_P \xi\, dx = \int_\Omega f_I \xi\, dx, \quad (2.12) \end{aligned}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, $\overline{s}_1 = \bar{f}^{-1}(\overline{p}_1 - \overline{p}_2)$ and $\overline{s}_2 = 1 - \bar{f}^{-1}(\overline{p}_1 - \overline{p}_2)$. The mobilities$M_1$, M_2 and the capillary function f are the extended functions on $\mathbb{R}$. After the passage to the limit in ε, we establish a maximum principle on saturations and then extended functions operate only on $[0, 1]$ where they have a physical meaning.
The system (2.11) – (2.12) can be written under the form

$$B_1(p_1, \varphi) = f_1(\varphi), \;\; B_2(p_2, \xi) = f_2(\xi), \quad (2.13)$$

where

$$B_1(p_1, \varphi) = \int_\Omega \mathbf{K} M_1^\varepsilon(\overline{s}_1) \rho_1(\overline{p}_1) \nabla p_1 \cdot \nabla \varphi\, dx,$$

$$\begin{aligned} f_1(\varphi) = - \int_\Omega \phi \frac{\rho_1(\overline{p}_1) Z(\overline{s}_1) - \rho_1^\star s_1^\star}{h} \varphi\, dx + \int_\Omega \mathbf{K} M_1(\overline{s}_1) \rho_1^2(\overline{p}_1) \mathbf{g} \cdot \nabla \varphi\, dx \\ - \int_\Omega \rho_1(\overline{p}_1) Z(\overline{s}_1) f_P \varphi\, dx - \eta \int_\Omega \rho_1(\overline{p}_1) \nabla(\mathcal{P}_N \overline{p}_1 - \mathcal{P}_N \overline{p}_2) \cdot \nabla \varphi\, dx, \end{aligned}$$

$$B_2(p_2, \xi) = \int_\Omega \mathbf{K} M_2^\varepsilon(\overline{s}_2) \nabla p_2 \cdot \nabla \xi\, dx,$$

$$f_2(\xi) = -\int_\Omega \phi \frac{Z(\overline{s}_2) - s_2^\star}{h} \xi \, dx + \int_\Omega \mathbf{K} M_2(\overline{s}_2) \rho_2 \mathbf{g} \cdot \nabla \xi \, dx \\ - \int_\Omega Z(\overline{s}_2) f_P \xi \, dx + \int_\Omega f_I \xi \, dx + \eta \int_\Omega \nabla(\mathcal{P}_N \overline{p}_1 - \mathcal{P}_N \overline{p}_2) \cdot \nabla \xi \, dx.$$

The maps $B_1(p_1, \varphi)$ and $B_2(p_2, \xi)$ are bilinear, continuous and coercive mappings on $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$. The coercivity is due to the presence of ε, and the assumptions (H2)-(H3) and (H5). The maps $f_1(\varphi)$ and $f_2(\xi)$ are linear continuous on $H^1_{\Gamma_1}(\Omega)$. Now, apply Lax-Milgram theorem to get the existence of the unique pair (p_1, p_2) in $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ for all $(\overline{p}_1, \overline{p}_2)$ belongs to $L^2(\Omega) \times L^2(\Omega)$.

Lemma 2.1. *The map $\mathcal{T}$ is a continuous operator which maps every bounded subsets of $L^2(\Omega)$ into a relatively compact set.*

Démonstration. Consider a sequence $(\overline{p}_{1,n}, \overline{p}_{2,n})$ of a bounded set of $L^2(\Omega) \times L^2(\Omega)$ which converges to $(\overline{p}_1, \overline{p}_2) \in L^2(\Omega) \times L^2(\Omega)$, and let us prove that $(p_{1,n}, p_{2,n}) = \mathcal{T}(\overline{p}_{1,n}, \overline{p}_{2,n})$ is bounded in $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ which converges to $(p_1, p_2) = \mathcal{T}(\overline{p}_1, \overline{p}_2)$. The sequences $p_{1,n}, \ p_{2,n}$ verify respectively

$$\int_\Omega \phi \frac{\rho_1(\overline{p}_{1,n}) Z(\overline{s}_{1,n}) - \rho_1^\star s_1^\star}{h} \varphi \, dx + \int_\Omega \mathbf{K} M_1^\varepsilon(\overline{s}_{1,n}) \rho_1(\overline{p}_{1,n}) \nabla p_{1,n} \cdot \nabla \varphi \, dx \\ - \int_\Omega \mathbf{K} M_1(\overline{s}_{1,n}) \rho_1^2(\overline{p}_{1,n}) \mathbf{g} \cdot \nabla \varphi \, dx + \eta \int_\Omega \rho_1(\overline{p}_{1,n}) \nabla(\mathcal{P}_N \overline{p}_{1,n} - \mathcal{P}_N \overline{p}_{2,n}) \cdot \nabla \varphi \, dx \\ + \int_\Omega \rho_1(\overline{p}_{1,n}) Z(\overline{s}_{1,n}) f_P \varphi \, dx = 0, \quad (2.14)$$

$$\int_\Omega \phi \frac{Z(\overline{s}_{2,n}) - s_2^\star}{h} \xi \, dx + \int_\Omega \mathbf{K} M_2^\varepsilon(\overline{s}_{2,n}) \nabla p_{2,n} \cdot \nabla \xi \, dx \\ - \int_\Omega \mathbf{K} M_2(\overline{s}_{2,n}) \rho_2 \mathbf{g} \cdot \nabla \xi \, dx - \eta \int_\Omega \nabla(\mathcal{P}_N \overline{p}_{1,n} - \mathcal{P}_N \overline{p}_{2,n}) \cdot \nabla \xi \, dx \\ + \int_\Omega Z(\overline{s}_{2,n}) f_P \xi \, dx = \int_\Omega f_I \xi \, dx, \quad (2.15)$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.
Let us take $\varphi = p_{1,n}$ in (2.14),

$$\int_\Omega \mathbf{K} M_1^\varepsilon(\overline{s}_{1,n}) \rho_1(\overline{p}_{1,n}) \nabla p_{1,n} \cdot \nabla p_{1,n} \, dx = \int_\Omega \mathbf{K} M_1(\overline{s}_{1,n}) \rho_1^2(\overline{p}_{1,n}) \mathbf{g} \cdot \nabla p_{1,n} \, dx \\ - \eta \int_\Omega \rho_1(\overline{p}_{1,n}) \nabla(\mathcal{P}_N \overline{p}_{1,n} - \mathcal{P}_N \overline{p}_{2,n}) \cdot \nabla p_{1,n} \, dx - \int_\Omega \phi \frac{\rho_1(\overline{p}_{1,n}) Z(\overline{s}_{1,n}) - \rho_1^\star s_1^\star}{h} p_{1,n} \, dx \\ - \int_\Omega \rho_1(\overline{p}_{1,n}) Z(\overline{s}_{1,n}) f_P p_{1,n} \, dx \quad (2.16)$$

we deduce from Cauchy-Schwartz inequality that (2.16) reduces to,

$$\begin{aligned}&\varepsilon k_0\rho_m\int_\Omega|\nabla p_{1,n}|^2\,dx\leq\\&\quad C(1+\|p_{1,n}\|_{L^2(\Omega)}+\|\nabla p_{1,n}\|_{L^2(\Omega)}+\|\nabla\mathcal{P}_N\overline{p}_{1,n}\|_{L^2(\Omega)}+\|\nabla\mathcal{P}_N\overline{p}_{2,n}\|_{L^2(\Omega)})\end{aligned}\tag{2.17}$$

where C depends on Ω, η, h, ϕ_1, $\|f_P\|_{L^2(\Omega)}$, $\|f_I\|_{L^2(\Omega)}$, ρ_M, k_∞ and $\|\rho_1^\star s_1^\star\|_{L^2(\Omega)}$. As,

$$\|\nabla\mathcal{P}_N\overline{p}_{i,n}\|_{L^2(\Omega)}\leq c_N\|\overline{p}_{i,n}\|_{L^2(\Omega)},\quad(i=1,2)$$

where c_N is the square root of the n^{th} eigenvalue of the Laplace operator (by considering the set of eigenvalues as increasing sequence), the Poincaré and Young inequalities and the estimate (2.17) ensure that the sequence $(p_{1,n})_n$ is uniformly bounded in $H^1_{\Gamma_1}(\Omega)$.
Then, taking $\xi=p_{2,n}$ in (2.15), we deduce similarly that, the sequence $(p_{2,n})_n$ is uniformly bounded in $H^1_{\Gamma_1}(\Omega)$. This establishes the relative compactness property of the map $\mathcal{T}$.

Furthermore, up to a subsequence, we have the convergences

$$p_{1,n}\longrightarrow p_1\qquad\text{weakly in } H^1_{\Gamma_1}(\Omega),\tag{2.18}$$

$$p_{2,n}\longrightarrow p_2\qquad\text{weakly in } H^1_{\Gamma_1}(\Omega),\tag{2.19}$$

$$p_{1,n}\longrightarrow p_1\qquad\text{strongly in } L^2(\Omega)\text{ and a.e. in }\Omega,\tag{2.20}$$

$$p_{2,n}\longrightarrow p_2\qquad\text{strongly in } L^2(\Omega)\text{ and a.e. in }\Omega.\tag{2.21}$$

In order to complete the proof of continuity of the operator $\mathcal{T}$, it is enough to show that (p_1,p_2) is the unique adherent value of the sequence $(p_{1,n},p_{2,n})$, for that let us show (p_1,p_2) is the unique solution of (2.11)-(2.12) by passing the limit in (2.14)-(2.15).

Passage to the limit in (2.14) :

$$\begin{aligned}&\int_\Omega\phi\frac{\rho_1(\overline{p}_{1,n})Z(\overline{s}_{1,n})-\rho_1^\star s_1^\star}{h}\varphi\,dx+\int_\Omega\mathbf{K}M_1^\varepsilon(\overline{s}_{1,n})\rho_1(\overline{p}_{1,n})\nabla p_{1,n}\cdot\nabla\varphi\,dx\\&-\int_\Omega\mathbf{K}M_1(\overline{s}_{1,n})\rho_1^2(\overline{p}_{1,n})\mathbf{g}\cdot\nabla\varphi\,dx+\eta\int_\Omega\rho_1(\overline{p}_{1,n})\nabla(\mathcal{P}_N\overline{p}_{1,n}-\mathcal{P}_N\overline{p}_{2,n})\cdot\nabla\varphi\,dx\\&+\int_\Omega\rho_1(\overline{p}_{1,n})Z(\overline{s}_{1,n})f_P\varphi\,dx=0,\end{aligned}$$

where $\overline{s}_{1,n}=\bar{f}^{-1}(\overline{p}_{1,n}-\overline{p}_{2,n})$.

The passage to the limit in the first term is due to the continuity of Z, $\bar{f}^{-1}$ and ρ_1, the convergences (2.20) and (2.21), and the domination of $\rho_1(\overline{p}_{1,n})Z(\overline{s}_{1,n})\varphi$ by

$\rho_M|\varphi|$, which allow us to apply the Lebesgue theorem.
The second term is treated as follows, the sequence $\left(KM_1^\varepsilon(\overline{s}_{1n})\rho_1(\overline{p}_{1n})\nabla\varphi\right)_n$ is dominated and converges a.e. as n goes to infinity. Then, by Lebesgue's theorem, we have the following strong convergence in $L^2(\Omega)$,

$$KM_1^\varepsilon(\overline{s}_{1,n})\rho_1(\overline{p}_{1n})\nabla\varphi \longrightarrow KM_1^\varepsilon(\overline{s}_1)\rho_1(\overline{p}_1)\nabla\varphi. \tag{2.22}$$

And, using the weak convergence (2.18), we establish the limit for the second term.
The fourth term

$$\eta\int_\Omega \rho_1(\overline{p}_{1,n})\nabla(\mathcal{P}_N\overline{p}_{1,n} - \mathcal{P}_N\overline{p}_{2,n})\cdot\nabla\varphi\,dx,$$

is treated as follows, we have

$$\rho_1(\overline{p}_{1,n})\nabla\varphi \longrightarrow \rho_1(\overline{p}_1)\nabla\varphi \text{ strongly in } (L^2(\Omega))^d. \tag{2.23}$$

Furthermore $\overline{p}_{i,n}$ converges in $L^2(\Omega)$, it follows that

$$\nabla\mathcal{P}_N\overline{p}_{i,n} \longrightarrow \nabla\mathcal{P}_N\overline{p}_i \text{ strongly in } (L^2(\Omega))^d \quad (i=1,2). \tag{2.24}$$

Then, the convergences (2.23)(2.24) allow us to pass the limit in the fourth term. The convergences of the other terms are always an application of the Lebesgue convergence theorem.
The passage to the limit on (2.15) is obtained in the same manner. Thus (p_1,p_2) is a solution of (2.11)-(2.12), which establishes the continuity and achieves the proof of the lemma. □

Lemma 2.2. *(A priori estimate) There exists $r>0$ such that, if $(p_1,p_2) = \lambda\mathcal{T}(p_1,p_2)$ with $\lambda\in(0,1)$, then*

$$\|(p_1,p_2)\|_{L^2(\Omega)\times L^2(\Omega)} \le r.$$

Démonstration. Assume $(p_1,p_2)=\lambda\mathcal{T}(p_1,p_2)$ exists , then (p_1,p_2) satisfies

$$\begin{aligned}\int_\Omega \mathbf{K}M_1^\varepsilon(s_1)\rho_1(p_1)\nabla p_1\cdot\nabla\varphi\,dx &= -\lambda\int_\Omega \phi\frac{\rho_1(p_1)Z(s_1)-\rho_1^\star s_1^\star}{h}\varphi\,dx \\ +\lambda\int_\Omega \mathbf{K}M_1(s_1)\rho_1^2(p_1)\mathbf{g}\cdot\nabla\varphi\,dx &- \lambda\int_\Omega \rho_1(p_1)Z(s_1)f_P\varphi\,dx \\ -\lambda\eta\int_\Omega \rho_1(p_1)\nabla(\mathcal{P}_Np_1-\mathcal{P}_Np_2)&\cdot\nabla\varphi\,dx,\end{aligned} \tag{2.25}$$

$$\int_\Omega \mathbf{K} M_2^\varepsilon(s_2)\nabla p_2\cdot\nabla\xi\,dx = -\lambda\int_\Omega \phi\frac{Z(s_2)-s_2^\star}{h}\xi\,dx \tag{2.26}$$
$$+\lambda\int_\Omega \mathbf{K} M_2(s_2)\rho_2\mathbf{g}\cdot\nabla\xi\,dx-\lambda\int_\Omega Z(s_2)f_P\xi\,dx+\lambda\int_\Omega f_I\xi\,dx$$
$$+\lambda\eta\int_\Omega \nabla(\mathcal{P}_Np_1-\mathcal{P}_Np_2)\cdot\nabla\xi\,dx.$$

for all (φ,ξ) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$.
Consider $\varphi=g_1(p_1):=\int_0^{p_1}\frac{1}{\rho_1(\zeta)}\,d\zeta\in H^1_{\Gamma_1}(\Omega)$ in (2.25) and $\xi=p_2\in H^1_{\Gamma_1}(\Omega)$ in (2.26). Summing these quantities, we obtain

$$\lambda\int_\Omega \frac{\phi}{h}\Big(\big(\rho_1(p_1)Z(s_1)-\rho_1^\star s_1^\star\big)g_1(p_1)+\big(Z(s_2)-s_2^\star\big)p_2\Big)\,dx \tag{2.27}$$
$$+\int_\Omega KM_1^\varepsilon\nabla p_1\cdot\nabla p_1\,dx+\lambda\eta\int_\Omega\nabla(\mathcal{P}_Np_1-\mathcal{P}_Np_2)\cdot\nabla(p_1-p_2)\,dx$$
$$-\lambda\int_\Omega K\rho_1(p_1)M_1(s_1)\mathbf{g}\cdot\nabla p_1\,dx+\int_\Omega KM_2^\varepsilon\nabla p_2\cdot\nabla p_2\,dx$$
$$+\lambda\int_\Omega\big(\rho_1(p_1)Z(s_1)g_1(p_1)+Z(s_2)p_2\big)f_P\,dx$$
$$-\lambda\int_\Omega K\rho_2M_2(s_2)\mathbf{g}\cdot\nabla p_2\,dx=\lambda\int_\Omega p_2f_I\,dx.$$

Remark that the functions $p_1\to g_1(p_1)$ is sub-linear, we deduce from Cauchy-Schwarz and Poincaré inequalities that (2.27) reduces to

$$\varepsilon\int_\Omega|\nabla p_1|^2\,dx+\varepsilon\int_\Omega|\nabla p_2|^2\,dx+\lambda\eta\int_\Omega|\nabla(\mathcal{P}_Np_1-\mathcal{P}_Np_2)|^2\,dx$$
$$\le C_1(1+\|f_P\|^2_{L^2(\Omega)}+\|f_I\|^2_{L^2(\Omega)}+\|\rho_1^\star s_1^\star\|^2_{L^2(\Omega)}+\|s_2^\star\|^2_{L^2(\Omega)}), \tag{2.28}$$

where C_1 depends on ε and not on λ. □

Lemma 2.1, Lemma 2.2 allow to apply the Leray-Schauder fixed point theorem [66], thus the proof of proposition 2.1 is finished. □

Step 2. Now we are concerned with the limit N goes to infinity (we omit the dependence of solutions on ε). For all N, we have established a solution $(p_{1,N},p_{2,N})\in H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$ to (2.9) (2.10) satisfying

$$\int_\Omega \phi \frac{\rho_1(p_1^N)Z(s_1^N) - \rho_1^\star s_1^\star}{h}\varphi\, dx + \int_\Omega \mathbf{K}M_1^\varepsilon(s_1^N)\rho_1(p_1^N)\nabla p_1^N \cdot \nabla\varphi\, dx$$
$$- \int_\Omega \mathbf{K}M_1(s_1^N)\rho_1^2(p_1^N)\mathbf{g}\cdot\nabla\varphi\, dx + \eta\int_\Omega \rho_1(p_1^N)\nabla(\mathcal{P}_N p_1^N - \mathcal{P}_N p_2^N)\cdot\nabla\varphi\, dx$$
$$+ \int_\Omega \rho_1(p_1^N)Z(s_1^N)f_P\varphi\, dx = 0, \quad (2.29)$$

$$\int_\Omega \phi\frac{Z(s_2^N) - s_2^\star}{h}\xi\, dx + \int_\Omega \mathbf{K}M_2^\varepsilon(s_2^N)\nabla p_2^N\cdot\nabla\xi\, dx$$
$$- \int_\Omega \mathbf{K}M_2(s_2^N)\rho_2\mathbf{g}\cdot\nabla\xi\, dx - \eta\int_\Omega \nabla(\mathcal{P}_N p_1^N - \mathcal{P}_N p_2^N)\cdot\nabla\xi\, dx$$
$$+ \int_\Omega Z(s_2^N)f_P\xi\, dx = \int_\Omega f_I\xi\, dx, \quad (2.30)$$

for all (φ,ξ) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$.
Reproducing the estimate (2.28) with $\lambda = 1$, we get

$$\varepsilon\int_\Omega |\nabla p_1^N|^2\, dx + \varepsilon\int_\Omega |\nabla p_2^N|^2\, dx + \eta\int_\Omega |\nabla(\mathcal{P}_N p_1^N - \mathcal{P}_N p_2^N)|^2\, dx$$
$$\leq C_1(1 + \|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)} + \|\rho_1^\star s_1^\star\|^2_{L^2(\Omega)} + \|s_2^\star\|^2_{L^2(\Omega)}), \quad (2.31)$$

where C_1 depends on ε and not on N.
Then, up to a subsequence, we have the convergences,

$$p_1^N \longrightarrow p_1 \text{ weakly in } H^1_{\Gamma_1}(\Omega), \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega \quad (2.32)$$
$$p_2^N \longrightarrow p_2 \text{ weakly in } H^1_{\Gamma_1}(\Omega), \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \quad (2.33)$$

The convergences in (2.29)-(2.30) with respect to N are obtained in the same manner as for the convergences with respect to n in (2.14) (2.15).

Step 3. Passage to the limit as ε goes to zero. For all $\varepsilon > 0$, we have shown that there exists $(p_{1,\varepsilon}, p_{2,\varepsilon}) \in H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$, satisfying

$$\int_\Omega \phi\frac{\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon) - \rho_1^\star s_1^\star}{h}\varphi\, dx + \int_\Omega \mathbf{K}M_1^\varepsilon(s_1^\varepsilon)\rho_1(p_1^\varepsilon)\nabla p_1^\varepsilon\cdot\nabla\varphi\, dx$$
$$-\int_\Omega \mathbf{K}M_1(s_1^\varepsilon)\rho_1^2(p_1^\varepsilon)\mathbf{g}\cdot\nabla\varphi\, dx + \eta\int_\Omega \rho_1(p_1^\varepsilon)\nabla(p_1^\varepsilon - p_2^\varepsilon)\cdot\nabla\varphi\, dx + \int_\Omega \rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)f_P\varphi\, dx = 0,$$
$$(2.34)$$

$$\int_\Omega \phi \frac{Z(s_2^\varepsilon) - s_2^\star}{h}\xi\, dx + \int_\Omega \mathbf{K} M_2^\varepsilon(s_2^\varepsilon)\nabla p_2^\varepsilon \cdot \nabla \xi\, dx$$
$$- \int_\Omega \mathbf{K} M_2(s_2^\varepsilon)\rho_2 \mathbf{g}\cdot\nabla\xi\, dx - \eta\int_\Omega \nabla(p_1^\varepsilon - p_2^\varepsilon)\cdot\nabla\xi\, dx + \int_\Omega Z(s_2^\varepsilon) f_P \xi\, dx = \int_\Omega f_I \xi\, dx, \tag{2.35}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.

Here we are going to use the feature of global pressure and the function $\beta(s_1^\varepsilon)$ (defined in (1.8)) to derive uniform estimates with respect to ε. After the passage to the limit in ε, a maximum principle on saturations is proven.

We state the following two lemmas in order to pass to the limit in ε.

Lemma 2.3. *The sequences $(s_i^\varepsilon)_\varepsilon$, $(p^\varepsilon := p_2^\varepsilon + \tilde{p}(s_1^\varepsilon))_\varepsilon$ defined by the proposition 2.1 satisfy*

$$(p^\varepsilon)_\varepsilon \text{ is uniformly bounded in } H^1_{\Gamma_1}(\Omega) \tag{2.36}$$
$$(\sqrt{\varepsilon}\, \nabla p_i^\varepsilon)_\varepsilon \text{ is uniformly bounded in } L^2(\Omega) \tag{2.37}$$
$$(\beta(s_1^\varepsilon))_\varepsilon \text{ is uniformly bounded in } H^1(\Omega) \tag{2.38}$$
$$(\nabla f(s_1^\varepsilon))_\varepsilon \text{ is uniformly bounded in } L^2(\Omega) \tag{2.39}$$

Démonstration. Consider $\varphi = g_1(p_1^\varepsilon) := \int_0^{p_1^\varepsilon} \frac{1}{\rho_1(\zeta)}\, d\zeta \in H^1_{\Gamma_1}(\Omega)$ in (2.34), $\xi = p_2^\varepsilon \in H^1_{\Gamma_1}(\Omega)$ in (2.35) and summing these quantities, we obtain

$$\int_\Omega \mathbf{K} M_1 \nabla p_1^\varepsilon\cdot\nabla p_1^\varepsilon\, dx + \int_\Omega \mathbf{K} M_2 \nabla p_2^\varepsilon\cdot\nabla p_2^\varepsilon\, dx + \varepsilon\int_\Omega \mathbf{K}\nabla p_1^\varepsilon\cdot\nabla p_1^\varepsilon\, dx \tag{2.40}$$
$$+\varepsilon\int_\Omega \mathbf{K}\nabla p_2^\varepsilon\cdot\nabla p_2^\varepsilon\, dx + \eta\int_\Omega \nabla f(s_1)\cdot\nabla f(s_1)\, dx = \int_\Omega \mathbf{K}\rho_1(p_1^\varepsilon) M_1(s_1^\varepsilon)\mathbf{g}\cdot\nabla p_1^\varepsilon\, dx$$
$$+ \int_\Omega \mathbf{K}\rho_2 M_2(s_2^\varepsilon)\mathbf{g}\cdot\nabla p_2^\varepsilon\, dx - \int_\Omega \Big(\rho_1(p_1^\varepsilon) Z(s_1^\varepsilon) g_1(p_1^\varepsilon) + Z(s_2^\varepsilon) p_2^\varepsilon\Big) f_P\, dx$$
$$+ \int_\Omega p_2^\varepsilon f_I\, dx - \int_\Omega \frac{\phi}{h}\Big(\Big(\rho_1(p_1^\varepsilon) Z(s_1^\varepsilon) - \rho_1^\star s_1^\star\Big) g_1(p_1^\varepsilon) + \Big(Z(s_2^\varepsilon) - s_2^\star\Big) p_2^\varepsilon\Big)\, dx.$$

To estimate the right hand side of (2.40), the Cauchy-Schwarz inequality combined with assumption (H2) leads to estimate the first two integrals

$$\left|\int_\Omega \mathbf{K}\rho_1(p_1^\varepsilon) M_1(s_1^\varepsilon)\mathbf{g}\cdot\nabla p_1^\varepsilon\, dx\right| \le C + \frac{k_0}{2}\int_\Omega M_1 \nabla p_1^\varepsilon\cdot\nabla p_1^\varepsilon\, dx, \tag{2.41}$$

$$\left|\int_\Omega \mathbf{K}\rho_2(p_2^\varepsilon) M_2(s_2^\varepsilon)\mathbf{g}\cdot\nabla p_2^\varepsilon\, dx\right| \le C + \frac{k_0}{2}\int_\Omega M_2 \nabla p_2^\varepsilon\cdot\nabla p_2^\varepsilon\, dx. \tag{2.42}$$

Now, the function $p_1 \to g_1(p_1)$ is sub-linear (i.e $|g_1(p_1)| \leq \frac{1}{\rho_m}|p_1|$) and,

$$\| p_1^\varepsilon \|_{L^2(\Omega)} \leq \| p^\varepsilon \|_{L^2(\Omega)} + \| \overline{p}(s_1^\varepsilon) \|_{L^2(\Omega)}$$

$$\| p_2^\varepsilon \|_{L^2(\Omega)} \leq \| p^\varepsilon \|_{L^2(\Omega)} + \| \tilde{p}(s_1^\varepsilon) \|_{L^2(\Omega)},$$

with the help of Poincaré inequality on the global pressure, it is possible to estimate the last three integrals by $C(1 + \int_\Omega M_1 \nabla p^\varepsilon \cdot \nabla p^\varepsilon \, dx)^{\frac{1}{2}}$.
For the left hand side of (2.40), we use the following key equation

$$\int_\Omega M(s_1^\varepsilon)|\nabla p^\varepsilon|^2 \, dx + \int_\Omega \frac{M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)}{M(s_1^\varepsilon)}|\nabla f(s_1^\varepsilon)|^2 \, dx \\ = \int_\Omega M_1(s_1^\varepsilon)\nabla p_1^\varepsilon \cdot \nabla p_1^\varepsilon \, dx + \int_\Omega M_2(s_2^\varepsilon)\nabla p_2^\varepsilon \cdot \nabla p_2^\varepsilon \, dx. \quad (2.43)$$

Combine the left and right hand side estimates of (2.40), we get

$$\int_\Omega M(s_1^\varepsilon)|\nabla p^\varepsilon|^2 \, dx + \int_\Omega \frac{M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)}{M(s_1^\varepsilon)}|\nabla f(s_1^\varepsilon)|^2 \, dx \\ + \eta \int_\Omega \nabla(f(s_1)) \cdot \nabla(f(s_1)) \, dx + \varepsilon \int_\Omega \nabla p_1^\varepsilon \cdot \nabla p_1^\varepsilon \, dx + \varepsilon \int_\Omega \nabla p_2^\varepsilon \cdot \nabla p_2^\varepsilon \, dx \leq C, \quad (2.44)$$

where C is a generic constant depends on a known terms and independent of ε, and this with the assumption (H3) ensure the estimate (2.36), the estimate (2.37) is established due to (2.44). For the estimate (2.38) we have

$$\int_\Omega \frac{M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)}{M(s_1^\varepsilon)}|\nabla f(s_1^\varepsilon)|^2 \, dx \leq C,$$

using the assumption (H3),

$$\int_\Omega M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)|\nabla f(s_1^\varepsilon)|^2 \, dx \leq C,$$

which implies that,

$$\int_\Omega |\nabla \beta(s_1^\varepsilon)|^2 \, dx = \int_\Omega \frac{M_1^2(s_1^\varepsilon)M_2^2(s_2^\varepsilon)}{M^2(s_1^\varepsilon)}|\nabla f(s_1^\varepsilon)|^2 \, dx \\ \leq \frac{1}{2} \int_\Omega M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)|\nabla f(s_1^\varepsilon)|^2 \, dx \leq C,$$

and this leads to the desired estimate (2.38).
The last estimate (2.39) is a consequence of (2.44), and this closes the proof of lemma. □

From the previous lemma, we deduce the following convergences.

Lemma 2.4. *(Strong and weak convergences)*
Up to a subsequence the sequence $(s_i^\varepsilon)_\varepsilon$, $(p^\varepsilon)_\varepsilon$, $(p_i^\varepsilon)_\varepsilon$ verify the following convergence

$$
\begin{aligned}
&p^\varepsilon \longrightarrow p && \textit{weakly in } H^1_{\Gamma_1}(\Omega), && (2.45)\\
&\beta(s_1^\varepsilon) \longrightarrow \beta(s_1) && \textit{weakly in } H^1(\Omega), && (2.46)\\
&p^\varepsilon \longrightarrow p && \textit{almost everywhere in } \Omega, && (2.47)\\
&\beta(s_1^\varepsilon) \longrightarrow \beta(s_1) && \textit{almost everywhere in } \Omega, && (2.48)\\
&Z(s_1^\varepsilon) \longrightarrow Z(s_1) && \textit{almost everywhere in } \Omega, && (2.49)\\
&Z(s_1^\varepsilon) \longrightarrow Z(s_1) && \textit{strongly in } L^2(\Omega), && (2.50)\\
&p_i^\varepsilon \longrightarrow p_i && \textit{almost everywhere in } \Omega. && (2.51)
\end{aligned}
$$

Démonstration. The weak convergences (2.45)–(2.46) follows from the uniform estimates (2.36) and (2.38) of lemma 2.3, while

$$
\begin{aligned}
&p^\varepsilon \longrightarrow p \text{ a. e. in } \Omega,\\
&\beta(s_1^\varepsilon) \longrightarrow \beta^\star \text{ a. e. in } \Omega
\end{aligned}
$$

is due to the compact injection of $H^1_{\Gamma_1}$ into $L^2(\Omega)$.
From (2.8), we have $\beta(s_1) := \beta(Z(s_1))$ for $s_1 \in [0,1]$ and β^{-1} is continuous,

$$Z(s_1^\varepsilon) \longrightarrow \beta^{-1}(\beta^\star)) := Z(s_1) \text{ a. e. in } \Omega,$$

while the Lebesgue theorem ensures the strong convergence (2.50).
From the definition (2.8), and the previous convergences, we deduce the almost everywhere convergence (2.48). The almost everywhere convergence (2.51) is a consequence of (2.47)–(2.49) and the fact that $\bar{p}(s_1) := \bar{p}(Z(s_1))$ for $s_1 \in [0,1]$, and this close the proof of the lemma. □

In order to achieve the proof of Theorem 2.3, it remains to pass to the limit as ε goes to zero in the formulations (2.34)(2.35) and a proof of a maximum principle on saturations.

For all test functions $(\varphi, \xi) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \phi \frac{\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon) - \rho_1^\star s_1^\star}{h}\varphi\, dx + \int_\Omega \mathbf{K}M_1^\varepsilon(s_1^\varepsilon)\rho_1(p_1^\varepsilon)\nabla p_1^\varepsilon \cdot \nabla\varphi\, dx$$
$$- \int_\Omega \mathbf{K}M_1(s_1^\varepsilon)\rho_1^2(p_1^\varepsilon)\mathbf{g}\cdot\nabla\varphi\, dx + \eta\int_\Omega \rho_1(p_1^\varepsilon)\nabla(p_1^\varepsilon - p_2^\varepsilon)\cdot\nabla\varphi\, dx$$
$$+ \int_\Omega \rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)f_P\varphi\, dx = 0,$$

$$\int_\Omega \phi \frac{Z(s_2^\varepsilon) - s_2^\star}{h}\xi\, dx + \int_\Omega \mathbf{K}M_2^\varepsilon(s_2^\varepsilon)\nabla p_2^\varepsilon \cdot \nabla\xi\, dx$$
$$- \int_\Omega \mathbf{K}M_2(s_2^\varepsilon)\rho_2\mathbf{g}\cdot\nabla\xi\, dx - \eta\int_\Omega \nabla(p_1^\varepsilon - p_2^\varepsilon)\cdot\nabla\xi\, dx + \int_\Omega Z(s_2^\varepsilon)f_P\xi\, dx = \int_\Omega f_I\xi\, dx,$$

The first terms of the above equality converge due to the strong convergence of $\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)$ to $\rho_1(p_1)Z(s_1)$ in $L^2(\Omega)$.
The second terms can be written as,

$$\int_\Omega \mathbf{K}M_1^\varepsilon(s_1^\varepsilon)\rho_1(p_1^\varepsilon)\nabla p_1^\varepsilon\cdot\nabla\varphi\, dx = \int_\Omega \mathbf{K}M_1(s_1^\varepsilon)\rho_1(p_1^\varepsilon)\nabla p^\varepsilon\cdot\nabla\varphi\, dx$$
$$+ \int_\Omega \mathbf{K}\rho_1(p_1^\varepsilon)\nabla\beta(s_1^\varepsilon)\cdot\nabla\varphi\, dx + \sqrt{\varepsilon}\int_\Omega \mathbf{K}\rho_1(p_1^\varepsilon)(\sqrt{\varepsilon}\,\nabla p_1^\varepsilon)\cdot\nabla\varphi\, dx. \quad (2.52)$$

The first two terms on the right hand side of the equation converge arguing in two steps. Firstly, the Lebesgue theorem and the convergences (2.49)(2.51) establish

$$\rho_1(p_1^\varepsilon)M_1(s_1^\varepsilon)\nabla\varphi \longrightarrow \rho_1(p_1)M_1(s_1)\nabla\varphi \text{ strongly in } (L^2(Q_T))^d,$$

$$\rho_1(p_1^\varepsilon)\nabla\varphi \longrightarrow \rho_1(p_1)\nabla\varphi \text{ strongly in } (L^2(Q_T))^d.$$

Secondly, the weak convergence on pressure (2.45) combined to the above strong convergence validate the convergence for the first term of the right hand side of (2.52), and the weak convergence (2.46) combined to the above strong convergence validate the convergence for the second term of the right hand side of (2.52).
The third term converges to zero due to the uniform estimate (2.37), and this achieves the passage to the limit on the second terms.
The convergences of the fourth terms of the above equations are due to the uniform estimate (2.39). The other terms converge using (2.49)(2.51) and the Lebesgue dominated convergence theorem. Similarly we can pass the limit to the other equality. So, there exists (p_1^h, p_2^h) solution of :

for all $\varphi, \xi \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \phi \frac{\rho_1(p_1^h)Z(s_1^h) - \rho_1^\star s_1^\star}{h} \varphi \, dx + \int_\Omega \mathbf{K} M_1(s_1^h)\rho_1(p_1^h)\nabla p_1^h \cdot \nabla\varphi \, dx \\ - \int_\Omega \mathbf{K} M_1(s_1^h)\rho_1^2(p_1^h)\mathbf{g} \cdot \nabla\varphi \, dx + \eta \int_\Omega \rho_1(p_1^h)\nabla(p_1^h - p_2^h) \cdot \nabla\varphi \, dx \\ + \int_\Omega \rho_1(p_1^h)Z(s_1^h) f_P \varphi \, dx = 0, \quad (2.53)$$

$$\int_\Omega \phi \frac{Z(s_2^h) - s_2^\star}{h} \xi \, dx + \int_\Omega \mathbf{K} M_2(s_2^h)\nabla p_2^h \cdot \nabla\xi \, dx - \int_\Omega \mathbf{K} M_2(s_2^h)\rho_2 \mathbf{g} \cdot \nabla\xi \, dx \\ - \eta \int_\Omega \nabla(p_1^h - p_2^h) \cdot \nabla\xi \, dx + \int_\Omega Z(s_2^h) f_P \xi \, dx = \int_\Omega f_I \xi \, dx. \quad (2.54)$$

Lemma 2.5. *$0 \le s_1^h,\ s_2^h \le 1$ a.e in Ω.*

Démonstration. It is enough to show that $s_i^h \ge 0$ a.e in Ω. For that, consider $\varphi = -(s_1)^-, \xi = -(s_2)^-$ respectively in (2.53) and (2.54), with $(s_1)^- = -\min(0, s_1) \ge 0$ and $s_1 = s_1^+ - (s_1)^-$. Taking into consideration the definition of the map Z, and according to the extension of the mobility of each phase, $M_i(s_i^h)(s_i^h)^- = 0 \quad (i = 1, 2.)$ we get

$$\int_\Omega \phi \frac{\rho_1^\star s_1^\star}{h}(s_1^h)^- \, dx + \eta \int_\Omega \rho_1(p_1^h)\bar{f}'(s_1^h)\nabla(s_1^h)^- \cdot \nabla(s_1^h)^- \, dx = 0, \quad (2.55)$$

$$\int_\Omega \phi \frac{s_2^\star}{h}(s_2^h)^- \, dx + \eta \int_\Omega \bar{f}'(s_1^h)\nabla(s_2^h)^- \cdot \nabla(s_2^h)^- \, dx = -\int_\Omega f_I (s_2^h)^- \, dx, \quad (2.56)$$

Since it is possible to choose an extension $\bar{f}$ of f out side $[0,1]$ in a way that ensures $\bar{f}'(s_1)$ different from zero out side $[0,1]$, we get

$$\eta \int_\Omega |\nabla(s_i^h)^-|^2 \, dx \le 0 \quad (i = 1, 2.),$$

which proves the maximum principle since s_2^- vanishes on Γ_1. □

After this maximum principle, the weak formulations (1.34) and (1.35) are established, and thus the theorem 2.3 is then established.

3 Proof of Theorem 2.2

The proof is based on a semi-discretization method in time [3]. Let be $T > 0$, $N \in \mathbb{N}^*$ and $h = \frac{T}{N}$. We define the following sequence parametrized by h :

$$p^0_{i,h}(x) = p^0_i(x) \ \text{ a.e. in } \Omega \qquad i = 1, 2, \tag{3.1}$$

for all $n \in [0, N-1]$, consider $(p^n_{1,h},\, p^n_{2,h}) \in L^2(\Omega) \times L^2(\Omega)$, denote by $(f_P)^{n+1}_h = \dfrac{1}{h}\displaystyle\int_{nh}^{(n+1)h} f_P(\tau)\, d\tau$ and $(f_I)^{n+1}_h = \dfrac{1}{h}\displaystyle\int_{nh}^{(n+1)h} f_I(\tau)\, d\tau$, then define $(p^{n+1}_{1,h},\, p^{n+1}_{2,h})$ solution of

$$\begin{aligned}\phi\frac{\rho_1(p^{n+1}_{1,h})s^{n+1}_{1,h} - \rho_1(p^n_{1,h})s^n_{1,h}}{h} &- \operatorname{div}(\mathbf{K}M_1(s^{n+1}_{1,h})\rho_1(p^{n+1}_{1,h})\nabla p^{n+1}_{1,h}) \\ &+ \operatorname{div}(\mathbf{K}\rho_1^2(p^{n+1}_{1,h})M_1(s^{n+1}_{1,h})\mathbf{g}) - \eta \operatorname{div}(\rho_1(p^{n+1}_{1,h})\nabla(p^{n+1}_{1,h} - p^{n+1}_{2,h})) \\ &+ \rho_1(p^{n+1}_{1,h})s^{n+1}_{1,h}(f_P)^{n+1}_h = 0,\end{aligned} \tag{3.2}$$

$$\begin{aligned}\phi\frac{s^{n+1}_{2,h} - s^n_{2,h}}{h} &- \operatorname{div}(\mathbf{K}M_2(s^{n+1}_{2,h})\nabla p^{n+1}_{2,h}) + \operatorname{div}(\mathbf{K}\rho_2 M_2(s^{n+1}_{2,h})\mathbf{g}) \\ &+ \eta \operatorname{div}(\nabla(p^{n+1}_{1,h} - p^{n+1}_{2,h})) + s^{n+1}_{2,h}(f_P)^{n+1}_h = (f_I)^{n+1}_h,\end{aligned} \tag{3.3}$$

with the boundary conditions (1.29). This sequence is well defined for all $n \in [0, N-1]$ by virtue of theorem 2.3. As a matter of fact, for given $s^n_{1,h}\rho_1(p^n_{1,h}) \geq 0$, $s^n_{2,h} \geq 0$ and $\rho_1(p^n_{1,h})s^n_{1,h} \in L^2(\Omega)$, $s^n_{2,h} \in L^2(\Omega)$, we construct $(p^{n+1}_{1,h}, p^{n+1}_{2,h}) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ so that $s^{n+1}_{i,h} \in [0,1]$.

Lemma 2.6. *(Uniform estimates with respect to h) The solutions of (3.2)-(3.3) satisfy*

$$\begin{aligned}\frac{1}{h}\int_\Omega \phi\Big(\mathcal{H}_1(p^{n+1}_{1,h})s^{n+1}_{1,h} - \mathcal{H}_1(p^n_{1,h})s^n_{1,h}\Big)\, dx &+ \frac{1}{h}\int_\Omega \phi\Big(\mathcal{F}(s^{n+1}_{1,h}) - \mathcal{F}(s^n_{1,h})\Big)\, dx \\ + \eta\int_\Omega |\nabla(p^{n+1}_{1,h} - p^{n+1}_{2,h})|^2\, dx &+ k_0\int_\Omega M_1(s^{n+1}_{1,h})\nabla p^{n+1}_{1,h}\cdot\nabla p^{n+1}_{1,h}\, dx \\ + k_0\int_\Omega M_2(s^{n+1}_{2,h})\nabla p^{n+1}_{2,h}\cdot\nabla p^{n+1}_{2,h}\, dx &\leq C(1 + \|(f_P)^{n+1}_h\|^2_{L^2(\Omega)} + \|(f_I)^{n+1}_h\|^2_{L^2(\Omega)}),\end{aligned} \tag{3.4}$$

where C does not depend on h,

$$\mathcal{H}_1(p_1) := \rho_1(p_1)g_1(p_1) - p_1, \quad \mathcal{F}(s) := \int_0^s f(\zeta)\, d\zeta \ \ \textit{and} \ \ g_1(p_1) = \int_0^{p_1}\frac{1}{\rho_1(\zeta)}\, d\zeta.$$

Démonstration. First of all, let us prove that : for all $s_i \geq 0$ and $s_i^\star \geq 0$ such that

$$s_1 + s_2 = s_1^\star + s_2^\star = 1,$$

$$\Big(\rho_1(p_1)s_1 - \rho_1(p_1^\star)s_1^\star\Big)g_1(p_1) + \Big(s_2 - s_2^\star\Big)p_2 \geq \mathcal{H}_1(p_1)s_1 - \mathcal{H}_1(p_1^\star)s_1^\star + \mathcal{F}(s_1) - \mathcal{F}(s_1^\star). \tag{3.5}$$

Let us denote by $\mathcal{J}$ the left hand side of (3.5),

$$\mathcal{J} = \Big(\rho_1(p_1)s_1 - \rho_1(p_1^\star)s_1^\star\Big)g_1(p_1) + \Big(s_2 - s_2^\star\Big)p_2.$$

Since the function g_1 is concave, we have

$$g_1(p_1) \leq g_1(p_1^\star) + g_1'(p_1^\star)(p_1 - p_1^\star).$$

From the definition of $\mathcal{H}_1$ and the concavity property of g_1, one gets

$$\begin{aligned}
\mathcal{J} &= \rho_1(p_1)s_1 g_1(p_1) - \rho_1(p_1^\star)s_1^\star g_1(p_1) + s_2 p_2 - s_2^\star p_2 \\
&\geq s_1\mathcal{H}_1(p_1) - s_1^\star\mathcal{H}_1(p_1^\star) \quad + s_1 p_1 - s_1^\star p_1 + s_2 p_2 - s_2^\star p_2 \\
&= s_1\mathcal{H}_1(p_1) - s_1^\star\mathcal{H}_1(p_1^\star) \quad + s_1\Big(p_1 - p_2\Big) - s_1^\star\Big(p_1 - p_2\Big) \\
&= s_1\mathcal{H}_1(p_1) - s_1^\star\mathcal{H}_1(p_1^\star) \quad + \Big(s_1 - s_1^\star\Big)f(s_1).
\end{aligned} \tag{3.6}$$

Since the function $\mathcal{F}$ is convex, then

$$(s_1 - s_1^\star)f(s_1) \geq \mathcal{F}(s_1) - \mathcal{F}(s_1^\star). \tag{3.7}$$

The above inequalities (3.6) and (3.7) verify that the assertion (3.5) is satisfied. Let us multiply scalarly (3.2) with $g_1(p_{1,h}^{n+1})$ and add the scalar product of (3.3) with $p_{2,h}^{n+1}$, we have

$$\begin{aligned}
&\frac{1}{h}\int_\Omega \phi\Big(\Big(\rho_1(p_{1,h}^{n+1})s_{1,h}^{n+1} - \rho_1(p_{1,h}^{n})s_{1,h}^{n}\Big)g_1(p_{1,h}^{n+1}) + \Big(s_{2,h}^{n+1} - s_{2,h}^{n}\Big)p_{2,h}^{n+1}\Big)\,dx \\
&\quad + \int_\Omega \mathbf{K}M_1(s_{1,h}^{n+1})\nabla p_{1,h}^{n+1}\cdot\nabla p_{1,h}^{n+1}\,dx + \int_\Omega \mathbf{K}M_2(s_{2,h}^{n+1})\nabla p_{2,h}^{n+1}\cdot\nabla p_{2,h}^{n+1}\,dx \\
&\quad + \eta\int_\Omega |\nabla f(s_{1,h}^{n+1})|^2\,dx = \int_\Omega \mathbf{K}M_1(s_{1,h}^{n+1})\rho_1(p_{1,h}^{n+1})\mathbf{g}\cdot\nabla p_{1,h}^{n+1}\,dx \\
&\quad + \int_\Omega \mathbf{K}M_2(s_{2,h}^{n+1})\rho_2\mathbf{g}\cdot\nabla p_{2,h}^{n+1}\,dx - \int_\Omega \rho_1(p_{1,h}^{n+1})s_{1,h}^{n+1}(f_P)_h^{n+1}g_1(p_{1,h}^{n+1})\,dx \\
&\quad - \int_\Omega s_{2,h}^{n+1}(f_P)_h^{n+1}p_{2,h}^{n+1}\,dx + \int_\Omega (f_I)_h^{n+1}p_{2,h}^{n+1}\,dx.
\end{aligned} \tag{3.8}$$

By a similar demonstration as *lemma* 2.3 one gets,

$$\frac{1}{h}\int_\Omega \phi\Big(\mathcal{H}_1(p_{1,h}^{n+1})s_{1,h}^{n+1} - \mathcal{H}_1(p_{1,h}^{n})s_{1,h}^{n}\Big)\,dx + \frac{1}{h}\int_\Omega \phi\Big(\mathcal{F}(s_{1,h}^{n+1}) - \mathcal{F}(s_{1,h}^{n})\Big)\,dx$$
$$+\eta\int_\Omega |\nabla(p_{1,h}^{n+1} - p_{2,h}^{n+1})|^2\,dx + k_0\int_\Omega M_1(s_{1,h}^{n+1})\nabla p_{1,h}^{n+1}\cdot\nabla p_{1,h}^{n+1}\,dx \tag{3.9}$$
$$+k_0\int_\Omega M_2(s_{2,h}^{n+1})\nabla p_{2,h}^{n+1}\cdot\nabla p_{2,h}^{n+1}\,dx \le C(1 + \|(f_P)_h^{n+1}\|^2_{L^2(\Omega)} + \|(f_I)_h^{n+1}\|^2_{L^2(\Omega)}),$$

and this achieves the proof of lemma 2.6. □

For a given sequence $(u_h^n)_n$, let us denote

$$\begin{aligned} &u_h(0) = u_h^0,\\ &u_h(t) = \sum_{n=0}^{N-1} u_h^{n+1}\chi_{]nh,(n+1)h]}(t), \quad \forall t \in]0,T] \end{aligned} \tag{3.10}$$

and

$$\tilde{u}_h(t) = \sum_{n=0}^{N-1}\Big((1+n-\frac{t}{h})u_h^n + (\frac{t}{h} - n)u_h^{n+1}\Big)\chi_{[nh,(n+1)h]}(t), \quad \forall t \in [0,T]. \tag{3.11}$$

Then,

$$\partial_t\tilde{u}_h(t) = \frac{1}{h}\sum_{n=0}^{N-1}(u_h^{n+1} - u_h^n)\chi_{]nh,(n+1)h[}(t), \quad \forall t \in [0,T]\backslash\{\cup_{n=0}^{N} nh\}$$

Let the functions $p_{i,h}$ and $s_{i,h}$ be defined as in (3.10) for $i = 1,2$. We denote by $r_{1,h}$ the function defined by (3.10) corresponding to $r_{1,h}^n = \rho_1(p_{1,h}^n)s_{1,h}^n$ and $\tilde{r}_{1,h}$ the function defined by (3.11) corresponding to $r_{1,h}^n$. In the same way, we denote by $f_{P,h}$ and $f_{I,h}$ the functions corresponding to $(f_P)_h^{n+1}$ and $(f_I)_h^{n+1}$ respectively.
Proposition 2.2. *Suppose the initial conditions p_1^0, s_1^0 being in $H^1(\Omega)$. The se-*

quence

$$(p_{i,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_1}(\Omega)), i=1,2, \tag{3.12}$$
$$(f(s_{1,h}))_h \text{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_1}(\Omega)), \tag{3.13}$$
$$(s_{1,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_1}(\Omega)), \tag{3.14}$$
$$(r_{1,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega)), \tag{3.15}$$
$$(\tilde{r}_{1,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega)), \tag{3.16}$$
$$(\tilde{s}_{2,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega)), \tag{3.17}$$
$$(\phi\partial_t\tilde{s}_{2,h})_h \text{ is uniformly bounded in } L^2(0,T;(H^1_{\Gamma_1}(\Omega))'), \tag{3.18}$$
$$(\phi\partial_t\tilde{r}_{1,h})_h \text{ is uniformly bounded in } L^2(0,T;(H^1_{\Gamma_1}(\Omega))'). \tag{3.19}$$

Démonstration. At the beginning of this proof, we indicate to some useful remarks which can be established by a classical calculations,

$$\int_{Q_T} M_i(s_{i,h})|\nabla p_{i,h}|^2\,dxdt = h\sum_{n=0}^{N-1}\int_\Omega M_i(s_{i,h}^{n+1})|\nabla p_{i,h}^{n+1}|^2\,dx \quad (i=1,2), \tag{3.20}$$
$$\int_{Q_T} |\nabla f(s_{1,h})|^2\,dxdt = h\sum_{n=0}^{N-1}\int_\Omega |\nabla f(s_{1,h}^{n+1})|^2\,dx, \tag{3.21}$$
$$\int_{Q_T} |f_P(t,x)|^2\,dtdx \geq h\sum_{n=0}^{N-1} \|(f_P)_h^{n+1}\|^2_{L^2(\Omega)}, \tag{3.22}$$
$$\int_{Q_T} |f_I(t,x)|^2\,dtdx \geq h\sum_{n=0}^{N-1} \|(f_I)_h^{n+1}\|^2_{L^2(\Omega)}. \tag{3.23}$$

Now, multiply (3.4) by h and summing it from $n=0$ to $n=N-1$,

$$\begin{aligned}\int_\Omega \phi\mathcal{H}_1(p_{1,h}(T))s_{1,h}(T)\,dx &+ k_0\int_{Q_T} M_1(s_{1,h})|\nabla p_{1,h}|^2\,dxdt\\
&+ k_0\int_{Q_T} M_2(s_{2,h})|\nabla p_{2,h}|^2\,dxdt + \eta\int_{Q_T}|\nabla f(s_{1,h})|^2\,dxdt\\
&\leq \int_\Omega\Big(\phi\mathcal{H}_1(p_{1,h}(0))s_{1,h}(0)\Big)dx + \mathcal{F}(s_{1,h}(0))\\
&\quad - \mathcal{F}(s_{1,h}(T)) + C\Big(1+\|f_P\|^2_{L^2(Q_T)} + \|f_I\|^2_{L^2(Q_T)}\Big),\end{aligned} \tag{3.24}$$

where C is a constant independent of h.
The positivity of the first term on the left hand side of (3.24) ensures that,

$$\begin{aligned}k_0\int_{Q_T} M_1(s_{1,h})|\nabla p_{1,h}|^2\,dxdt &+ k_0\int_{Q_T} M_2(s_{2,h})|\nabla p_{2,h}|^2\,dxdt\\
&+ \eta\int_{Q_T}|\nabla f(s_{1,h})|^2\,dxdt < C,\end{aligned} \tag{3.25}$$

since we have,

$$\int_{Q_T} M_1(s_{1,h})|\nabla p_{1,h}|^2\,dxdt + \int_{Q_T} M_2(s_{2,h})|\nabla p_{2,h}|^2\,dxdt$$
$$= \int_{Q_T} M(s_{1,h})|\nabla p_h|^2\,dxdt + \int_{Q_T} \frac{M_1(s_{1,h})M_2(s_{2,h})}{M(s_{1,h})}|\nabla f(s_{1,h})|^2\,dxdt, \quad (3.26)$$

then, one gets the main estimate :

$$\int_{Q_T} M(s_{1,h})|\nabla p_h|^2\,dxdt + \eta \int_{Q_T} |\nabla f(s_{1,h})|^2\,dxdt \le C, \quad (3.27)$$

which gives the estimate (3.13). For the second estimate (3.14) and first of all, let us indicate to the fact that,

$$p_{1,h}(t,x) - p_{2,h}(t,x) = 0 = f(s_{1,h}(t,x)) \qquad \text{for } \ x \in \Gamma_1$$

which gives that $s_{1,h/\Gamma_1} = 0$. The assumption (H6) on the capillary function f with the second term of (3.27) lead to

$$\int_{Q_T} |\nabla s_{1,h}|^2\,dxdt < C,$$

where C is a constant independent of h, and this achieves (3.14).

Since we have,

$$\nabla p_{1,h} = \nabla p_h + \frac{M_2}{M}\nabla f(s_{1,h}) \ \text{ and } \quad \nabla p_{2,h} = \nabla p_h - \frac{M_1}{M}\nabla f(s_{1,h}),$$

then, the estimate (3.12) becomes a consequence of (3.27).
The uniform estimate (3.15) is a consequence of the two previous ones since the density ρ_1 is bounded and of class $\mathcal{C}^1$ functions as well as the saturations $0 \le s_{i,h} \le 1$,

$$\nabla r_{1,h} = \sum_{n=0}^{N-1} \Big(\rho_1'(p_{1,h}^{n+1}) s_{1,h}^{n+1} \nabla p_{1,h}^{n+1} + \rho_1(p_{1,h}^{n+1}) \nabla s_{1,h}^{n+1}\Big) \chi_{]nh,(n+1)h]}(t).$$

Now, for estimate (3.16) we have,

$$\nabla \tilde{r}_{1,h} = \sum_{n=0}^{N-1} \Big((1+n-\frac{t}{h})[\rho_1'(p_{1,h}^{n}) s_{1,h}^{n} \nabla p_{1,h}^{n} + \rho_1(p_{1,h}^{n}) \nabla s_{1,h}^{n}]$$
$$+ (\frac{t}{h}-n)[\rho_1'(p_{1,h}^{n+1}) s_{1,h}^{n+1} \nabla p_{1,h}^{n+1} + \rho_1(p_{1,h}^{n+1}) \nabla s_{1,h}^{n+1}]\Big) \chi_{]nh,(n+1)h]}(t). \quad (3.28)$$

since the density ρ_1 is bounded and of class $\mathcal{C}^1$ function as well as the saturations $0 \le s^n_{i,h} \le 1$,

$$|\nabla \tilde{r}_{1,h}|^2 \le C \sum_{n=0}^{N-1} \Big(|\nabla p^n_{1,h}|^2 + |\nabla s^n_{1,h}|^2 + |\nabla p^{n+1}_{1,h}|^2 + |\nabla s^{n+1}_{1,h}|^2 \Big) \chi_{]nh,(n+1)h]}(t),$$

and this implies that,

$$||\nabla \tilde{r}_{1,h}||^2_{L^2(Q_T)} \le C(\|\nabla p^0_{1,h}\|^2_{L^2(\Omega)} + \|\nabla s^0_{1,h}\|^2_{L^2(\Omega)} + \|\nabla p_{1,h}\|^2_{L^2(Q_T)} + \|\nabla s_{1,h}\|^2_{L^2(Q_T)})$$

where C is a constant independent of h, and this achieve estimate (3.16).
From equations (3.2) and (3.3), we have for all $\varphi \in L^2(0,T; H^1_{\Gamma_1}(\Omega))$,

$$\begin{aligned} \langle \phi \partial_t \tilde{r}_{1,h}, \varphi \rangle &= - \int_{Q_T} \mathbf{K} M_1(s_{1,h}) \rho_1(p_{1,h}) \nabla p_{1,h} \cdot \nabla \varphi \, dxdt \\ &+ \int_{Q_T} \mathbf{K} \rho_i^2(p_{1,h}) M_1(s_{1,h}) \mathbf{g} \cdot \nabla \varphi \, dxdt - \eta \int_{Q_T} \nabla (p_{1,h} - p_{2,h}) \cdot \nabla \varphi \, dxdt \\ &- \int_{Q_T} \rho_1(p_{1,h}) s_{1,h} f_{P,h} \varphi \, dxdt, \end{aligned}$$

$$\begin{aligned} \langle \phi \partial_t \tilde{r}_{2,h}, \varphi \rangle &= - \int_{Q_T} \mathbf{K} M_2(s_{2,h}) \rho_2(p_{2,h}) \nabla p_{2,h} \cdot \nabla \varphi \, dxdt \\ &+ \int_{Q_T} \mathbf{K} \rho_i^2(p_{2,h}) M_2(s_{2,h}) \mathbf{g} \cdot \nabla \varphi \, dxdt + \eta \int_{Q_T} \nabla (p_{1,h} - p_{2,h}) \cdot \nabla \varphi \, dxdt \\ &- \int_{Q_T} \rho_2(p_{2,h}) s_{2,h} f_{P,h} \varphi \, dxdt + \int_{Q_T} \rho_2(p_{2,h}) f_{I,h} \varphi \, dxdt. \end{aligned}$$

The above estimates (3.14)–(3.12) with (3.27) ensure that $(\phi \partial_t \tilde{r}_{i,h})_h$ is uniformly bounded in $L^2(0,T; (H^1_{\Gamma_1}(\Omega))')$. $\square$

The next step is to pass from an elliptic problem to a parabolic one. Then, we pass to the limit on h, using some compactness theorems.

Proposition 2.3. *(Convergence with respect to h) We have the following conver-*

gences as h goes to zero,

$$\|r_{1,h} - \tilde{r}_{1,h}\|_{L^2(0,T;H^{\frac{1}{2}}(\Omega))} \longrightarrow 0, \tag{3.29}$$

$$\|s_{2,h} - \tilde{s}_{2,h}\|_{L^2(0,T;H^{\frac{1}{2}}(\Omega))} \longrightarrow 0, \tag{3.30}$$

$$p_{i,h} \longrightarrow p_i \text{ weakly in } L^2(0,T;H^1_{\Gamma_1}(\Omega)) \ (i=1,2.), \tag{3.31}$$

$$s_{2,h} \longrightarrow s_2 \text{ weakly in } L^2(0,T;H^1_{\Gamma_1}(\Omega)), \tag{3.32}$$

$$p_{i,h} \longrightarrow p_i \text{ weakly in } L^2(\Sigma_T), \tag{3.33}$$

$$s_{2,h} \longrightarrow s_2 \text{ strongly in } L^2(0,T;H^{\frac{1}{2}}(\Omega)), \tag{3.34}$$

$$s_{2,h} \longrightarrow s_2 \text{ strongly in } L^2(\Sigma_T), \tag{3.35}$$

$$r_{1,h} \longrightarrow r_1 \text{ strongly in } L^2(0,T;H^{\frac{1}{2}}(\Omega)). \tag{3.36}$$

$$r_{1,h} \longrightarrow r_1 \text{ strongly in } L^2(\Sigma_T). \tag{3.37}$$

Furthermore,

$$s_{i,h} \longrightarrow s_i \text{ almost everywhere in } Q_T, \tag{3.38}$$

$$0 \le s_i \le 1 \text{ almost everywhere in } Q_T, \tag{3.39}$$

and

$$r_1 = \rho_1(p_1)s_1 \text{ almost everywhere in } Q_T \text{ and almost everywhere in } \Sigma_T. \tag{3.40}$$

Finally, we have,

$$f_1(p_{1,h})f_2(s_{1,h}) \longrightarrow f_1(p_1)f_2(s_1) \text{ a.e. in } Q_T, \forall f_1, f_2 \in \mathcal{C}^0_b(\mathbb{R}) \text{ such that } f_2(0)=0, \tag{3.41}$$

$$\phi\partial_t\tilde{r}_{1,h} \longrightarrow \phi\partial_t(\rho(p_1)s_1) \text{ weakly in } L^2(0,T;(H^1_{\Gamma_w}(\Omega))') \tag{3.42}$$

$$\phi\partial_t\tilde{s}_{2,h} \longrightarrow \phi\partial_t s_2 \text{ weakly in } L^2(0,T;(H^1_{\Gamma_w}(\Omega))'). \tag{3.43}$$

Démonstration. Note that

$$\begin{aligned}\|r_{1,h} - \tilde{r}_{1,h}\|^2_{L^2(Q_T)} &= \sum_{n=0}^{N-1}\int_{nh}^{(n+1)h} \|((1+n-\frac{t}{h})(r^{n+1}_{1,h} - r^n_{1,h})\|^2_{L^2(\Omega)}\,dt \\ &= \frac{h}{3}\sum_{n=0}^{N-1}\|r^{n+1}_{1,h} - r^n_{1,h}\|^2_{L^2(\Omega)},\end{aligned}$$

$$
\begin{aligned}
\|s_{2,h} - \tilde{s}_{2,h}\|^2_{L^2(Q_T)} &= \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \|((1+n-\frac{t}{h})(s^{n+1}_{2,h} - s^n_{2,h})\|^2_{L^2(\Omega)}\, dt \\
&= \frac{h}{3} \sum_{n=0}^{N-1} \|s^{n+1}_{2,h} - s^n_{2,h}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

We multiply scalarly (3.2) and (3.3) respectively with $r^{n+1}_{1,h} - r^n_{1,h}$ and $s^{n+1}_{2,h} - s^n_{2,h}$. Then, summing for $n = 0$ to $N-1$, we get
for the water equation,

$$
\begin{aligned}
\frac{\phi_0}{h} \sum_{n=0}^{N-1} \|s^{n+1}_{2,h} - s^n_{2,h}\|^2_{L^2(\Omega)} \leq C \sum_{n=0}^{N-1} \Big(&\|\nabla s^n_{2,h}\|^2_{L^2(\Omega)} + \|\nabla s^{n+1}_{2,h}\|^2_{L^2(\Omega)} \\
&+ \|\nabla p^{n+1}_{2,h}\|^2_{L^2(\Omega)} + \|(f_P)^{n+1}_h\|^2_{L^2(\Omega)} + \|(f_I)^{n+1}_h\|^2_{L^2(\Omega)} \Big).
\end{aligned}
$$

This yields to

$$
\begin{aligned}
\sum_{n=0}^{N-1} \|s^{n+1}_{2,h} - s^n_{2,h}\|^2_{L^2(\Omega)} \leq C\Big(1 &+ \|\nabla s_{2,h}\|^2_{L^2(Q_T)} \\
&+ \|\nabla p_{2,h}\|^2_{L^2(Q_T)} + \|f_P\|^2_{L^2(Q_T)} + \|f_I\|^2_{L^2(Q_T)} \Big).
\end{aligned}
$$

And from (3.14),(3.12) and (3.15), we conclude that

$$
\|s_{2,h} - \tilde{s}_{2,h}\|_{L^2(Q_T)} \longrightarrow 0,
$$

Using again (3.14) and by interpolation argument, we obtain (3.30) since

$$
\|s_{2,h} - \tilde{s}_{2,h}\|_{L^2(0,T;H^{\frac{1}{2}}(\Omega))} \leq C \|s_{2,h} - \tilde{s}_{2,h}\|_{L^2(Q_T)} \|s_{2,h} - \tilde{s}_{2,h}\|_{L^2(0,T;H^1(\Omega))}.
$$

And, for the gas equation,

$$
\begin{aligned}
\frac{\phi_0}{h} \sum_{n=0}^{N-1} \|r^{n+1}_{1,h} - r^n_{1,h}\|^2_{L^2(\Omega)} \leq C \sum_{n=0}^{N-1} \Big(&\|\nabla r^n_{1,h}\|^2_{L^2(\Omega)} + \|\nabla r^{n+1}_{1,h}\|^2_{L^2(\Omega)} \\
&+ \|\nabla s^{n+1}_{2,h}\|^2_{L^2(\Omega)} + \|\nabla p^{n+1}_{1,h}\|^2_{L^2(\Omega)} + \|(f_P)^{n+1}_h\|^2_{L^2(\Omega)} \Big).
\end{aligned}
$$

This yields to

$$
\begin{aligned}
\sum_{n=0}^{N-1} \|r^{n+1}_{1,h} - r^n_{1,h}\|^2_{L^2(\Omega)} \leq C\Big(1 + \|\nabla r_{1,h}\|^2_{L^2(Q_T)} &+ \|\nabla s_{2,h}\|^2_{L^2(Q_T)} \\
&+ \|\nabla p_{1,h}\|^2_{L^2(Q_T)} + \|f_P\|^2_{L^2(Q_T)} \Big).
\end{aligned}
$$

And from (3.14),(3.12) and (3.15), we deduce that

$$\|r_{1,h} - \tilde{r}_{1,h}\|_{L^2(Q_T)} \longrightarrow 0,$$

In the same way as above, using (3.15), by interpolation we obtain (3.29) since,

$$\|r_{1,h} - \tilde{r}_{1,h}\|_{L^2(0,T;H^{\frac{1}{2}}(\Omega))} \leq C\|r_{1,h} - \tilde{r}_{1,h}\|_{L^2(Q_T)}\|r_{1,h} - \tilde{r}_{1,h}\|_{L^2(0,T;H^1(\Omega))}.$$

From (3.14)-(3.12) , the sequences $(s_{2,h})_h$, $(p_{i,h})_h$, are uniformly bounded in $L^2(0,T;H^1_{\Gamma_1}(\Omega))$, we have up to a subsequence the convergence result (3.31),(3.32). By virtue of the continuity of the trace operator we have,

$$\|p_{i,h}\|^2_{L^2(\partial\Omega)} \leq C\|p_{i,h}\|^2_{H^1(\Omega)}$$

To conclude the weak convergence (3.33), we integrate over $(0,T)$ the above inequality to deduce that $p_{i,h}$ is uniformly bounded in $L^2(\Sigma_T)$.
From (3.14), the sequence $(\tilde{s}_{2,h})_h$ is also uniformly bounded in $L^2(0,T;H^1_{\Gamma_1}(\Omega))$ and from estimate (3.18) we have up to a subsequence the convergence result

$$\tilde{s}_{2,h} \longrightarrow s_2 \text{ strongly in } L^2(0,T;H^{\frac{1}{2}}(\Omega)).$$

This compact result is classical and can be found in [62], [22] when the porosity is constant, and under the assumption (H1) (the porosity belongs to $W^{1,\infty}(\Omega)$), the proof can be adapted with minor modifications. By virtue of (3.30), we deduce (3.34) and (3.35).
The sequence $(\tilde{r}_{1,h})_h$ is uniformly bounded in $L^2(0,T;H^1_{\Gamma_w}(\Omega))$, (3.16), and in light of (3.19) we have the strong convergence

$$\tilde{r}_{1,h} \longrightarrow r_1 \text{ strongly in } L^2(0,T;H^{\frac{1}{2}}(\Omega)).$$

Then, (3.29) finishes to establish (3.36) and (3.37).
Recall that $r_{1,h} = \rho_1(p_{1,h})s_{1,h}$ and from the convergence (3.36)

$$\rho_1(p_{1,h})s_h \longrightarrow r_1 \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{3.44}$$

Let us show that $r_1 = \rho_1(p_1)s_1$.
We consider

$$\int_{Q_T} s_{1,h}(\rho_1(p_{1,h}) - \rho_1(v))(p_{1,h} - v)dtdx, \ \ \forall v \in L^2(Q_T),$$

this quantity is positive by monotony of ρ_1 and converges to

$$\int_{Q_T} (r_1 - s_1\rho_1(v))(p_1 - v)\, dtdx \geq 0, \;\; \forall v \in L^2(Q_T),$$

by virtue of (3.36) and (3.31). Then, we choose $v = p_1 + \delta w$ with $\delta \in]0,1]$, $w \in L^2(Q_T)$. So, divide by δ and let δ goes to zero,

$$\int_{Q_T} (r_1 - s_1\rho_1(p_1))w\, dtdx \geq 0 \;\forall w \in L^2(Q_T).$$

This clearly shows the identification (3.40) on Q_T. We can do the same computations by integrating on Σ_T instead of Q_T to establish the identity (3.40) on Σ_T. Remark that the convergence on

$$\int_{\Sigma_T} s_{1,h}(\rho_1(p_{1,h}) - \rho_1(v))(p_{1,h} - v) dtd\sigma \;\; \forall v \in L^2(\Sigma_T),$$

is obtained by (3.33) and (3.37).
To conclude the a.e. convergence (3.41),on one hand, when $s_{1,h} \to s_1 = 0$ a.e., $f_1(p_{1,h})f_2(s_{1,h}) \to 0 = f_1(p_1)f_2(s_1)$ a.e. (since $f_2(0) = 0$ and f_1 is bounded). On the other hand, when $s_{1,h} \to s_1 \neq 0$, in light of (3.44) we have $f_1(p_{1,h}) \to f_1(p_1)$ a.e.. Then, $f_1(p_{1,h})f_2(s_{1,h}) \to f_1(p_1)f_2(s_1)$ since f_1, f_2 are continuous and this establish (3.41).
Finally, the weak convergences (3.42) and (3.43) are a consequence of (3.18) and (3.19). In particular, identification of the limit for (3.42) is due to (3.40). □

The technique for obtaining solutions of the system (1.27)–(1.28) is to pass to the limit as h goes to zero on the solutions of

$$\begin{aligned}\phi\partial_t(\tilde{r}_{1,h}) - \operatorname{div}(\mathbf{K}M_1(s_{1,h})\rho_1(p_{1,h})\nabla p_{1,h}) + \operatorname{div}(\mathbf{K}M_1(s_{1,h})\rho_1^2(p_{1,h})\mathbf{g})\\ - \eta \operatorname{div}(\rho_1(p_{1,h})\nabla(p_{1,h} - p_{2,h})) + \rho_1(p_{1,h})s_{1,h}f_{P,h} = 0 \quad (3.45)\end{aligned}$$

$$\begin{aligned}\phi\partial_t(\tilde{s}_{2,h}) - \operatorname{div}(\mathbf{K}M_2(s_{2,h})\nabla p_{2,h}) + \operatorname{div}(\mathbf{K}M_2(s_{2,h})\rho_2\mathbf{g})\\ + \eta \operatorname{div}(\nabla(p_{1,h} - p_{2,h})) + s_{2,h}f_{P,h} = f_{I,h} \quad (3.46)\end{aligned}$$

Remark that this system is nothing else than (3.2)-(3.3), written for $n = 0$ to $N-1$ by using the definition (3.10) and (3.11). Let us consider the weak formulations $(i = 1, 2)$ on which we have to pass to the limit

$$\langle \phi\partial_t \tilde{r}_{1,h}, \varphi_1\rangle + \int_{Q_T} \mathbf{K} M_1(s_{1,h})\rho_1(p_{1,h})\nabla p_{1,h}\cdot\nabla\varphi_1\,dxdt$$
$$-\int_{Q_T} \mathbf{K}\rho_1^2(p_{1,h})M_1(s_{1,h})\mathbf{g}\cdot\nabla\varphi_1\,dxdt + \eta\int_{Q_T}\rho_1(p_{1,h})\nabla(p_{1,h}-p_{2,h})\cdot\nabla\varphi_1\,dxdt$$
$$+\int_{Q_T}\rho_1(p_{1,h})s_{1,h}f_{P,h}\varphi_1\,dxdt = 0. \quad (3.47)$$

$$\langle \phi\partial_t \tilde{s}_{2,h}, \varphi_2\rangle + \int_{Q_T} \mathbf{K} M_2(s_{2,h})\nabla p_{2,h}\cdot\nabla\varphi_2\,dxdt$$
$$-\int_{Q_T} \mathbf{K}\rho_2 M_2(s_{2,h})\mathbf{g}\cdot\nabla\varphi_2\,dxdt - \eta\int_{Q_T}\nabla(p_{1,h}-p_{2,h})\cdot\nabla\varphi_2\,dxdt$$
$$+\int_{Q_T} s_{2,h}f_{P,h}\varphi\,dxdt = \int_{Q_T} f_{I,h}\varphi_2\,dxdt. \quad (3.48)$$

where φ_1, φ_2 belong to $L^2(0,T;H^1_{\Gamma_1}(\Omega))$.
Next we pass to the limit on each term of (3.47)-(3.48) which is conserved by the previous proposition.
Consider firstly (3.47) :
The passage to the limit on the first term is due to (3.42). For the second term we have $M_1(s_{1,h})\rho_1(p_{1,h})\nabla\varphi$ converges almost everywhere in Q_T and dominated which leads by Lebesgue theorem to a strong convergence in $L^2(Q_T)$ and by virtue of the weak convergence (3.31) we establish the convergence of the second term of (3.47) to the desired term. The third and fifth terms converge obviously to the wanted limit due to the previous proposition and Lebesgue theorem.
The passage to the limit in fourth term is technical somehow. We first consider a smooth function φ_1^* approaching φ_1 in order to integrate by parts,

$$\int_{Q_T}\rho_1(p_{1,h})\nabla(p_{1,h}-p_{2,h})\cdot\nabla\varphi_1^*\,dxdt = \int_{Q_T}\rho_1(p_{1,h})\nabla\Big(f(s_{1,h})-f(0)\Big)\cdot\nabla\varphi_1^*\,dxdt \quad (3.49)$$
$$= -\int_{Q_T}\rho_1\Big(f(s_{1,h})-f(0)\Big)\,\mathrm{div}\,\nabla\varphi_1^*\,dxdt + \int_{\Sigma_T}\rho_1\Big(f(s_{1,h})-f(0)\Big)\nabla\varphi_1^*\cdot n\,d\sigma dt$$
$$-\int_{Q_T}\rho_1^{'}\Big(f(s_{1,h})-f(0)\Big)\nabla p_{1,h}\cdot\nabla\varphi_1^*\,dxdt.$$

It is now possible to pass to the limit on the right hand-side. The first term converges to the desired term by virtue of (3.41) with $f_1=\rho_1$ and $f_2 = f - f(0)$. The second term converges in the same way, but on the boundary. The last term converges to the desired limit by using again (3.41) with $f_1 = \rho_1^{'}$ and $f_2 = f - f(0)$ to apply the Lebesgue theorem on $\rho_1^{'}(p_{1,h})\Big(f(s_{1,h})-f(0)\Big)\nabla\varphi_1^*$. Then the weak convergence (3.31) allows to conclude.

After passing the limit in (3.49), we integrate again by parts to establish,

$$\int_{Q_T} \rho_1(p_{1,h})\nabla(p_{1,h} - p_{2,h}) \cdot \nabla\varphi_1^* \, dxdt \to \int_{Q_T} \rho_1(p_1)\nabla(p_1 - p_2) \cdot \nabla\varphi_1^* \, dxdt$$

Now, by density argument, this convergence exists with φ_1 instead of φ_1^*, i.e. as h goes to zero,

$$\eta \int_{Q_T} \rho_1(p_{1,h})\nabla(p_{1,h} - p_{2,h}) \cdot \nabla\varphi_1 \, dxdt \to \eta \int_{Q_T} \rho_1(p_1)\nabla(p_1 - p_2) \cdot \nabla\varphi_1 \, dxdt$$

Secondly, consider (3.48) :
The first term converges due to the weak convergence (3.43). The convergence of the second term is a consequence of the weak convergence (3.31) and a Lebesgue theorem application on $M_2(s_{2,h})\nabla\varphi_2$. The forth one follows from the weak convergence (3.31), and the convergences for the other terms is a simple application of Lebesgue theorem.

We then establish the weak formulation (1.30)-(1.31) of theorem 2.2. Furthermore, we have well obtained by proposition 2.3

$$\begin{aligned}
&0 \le s_i(t,x) \le 1 \text{ a.e. in } Q_T, \quad s_2 \in L^2(0,T; H^1_{\Gamma_1}(\Omega)), \\
&p_i \in L^2(0,T; H^1_{\Gamma_1}(\Omega))(i = 1, 2.),\ \rho_1(p_1)s_1 \in L^2(0,T; H^1(\Omega)), \\
&\phi\partial_t(\rho_1(p_1)s_1) \in L^2(0,T; (H^1_{\Gamma_1}(\Omega))'),\ \phi\partial_t s_2 \in L^2(0,T; (H^1_{\Gamma_1}(\Omega))')
\end{aligned}$$

The compactness property on $\rho_1(p_{1,h})s_{1,h}$ and $s_{2,h}$ implies $\rho_1(p_1)s_1$, s_2 belong to $C^0([0,T]; L^2(\Omega)))$, and this achieve the proof of Theorem 2.2 under the assumption on regular initial conditions. It is possible to regularize the initial conditions $p_i^0 \in L^2(\Omega)$ by $p_{i,\nu}^0 \in H^1(\Omega)$ such that $p_{i,\nu}^0 \to p_i^0$ strongly in $L^2(\Omega)$ when $\nu \to 0$, for $i = 1, 2$. Then we can reproduce energy estimates for the non-degenerate parabolic system independent of the parameter of regularization, since the estimates are depending only on $L^2(\Omega)$ initial conditions, and pass to the limit in this system to establish the proof of Theorem 2.2.

4 Proof of Theorem 2.1

The proof is based on the existence result established for the non-degenerate case and some compactness technique on the evolution terms.
Lemma 2.7. *The sequences $(s_i^\eta)_\eta$, $(p^\eta := p_2^\eta + \tilde{p}(s_1^\eta))_\eta$ defined by the Theorem 2.2*

satisfy

$$0 \le s_i^\eta(t,x) \le 1 \quad a.e. \quad in \quad (t,x) \in Q_T \tag{4.1}$$
$$(p^\eta)_\eta \text{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_1}(\Omega)) \tag{4.2}$$
$$(\sqrt{\eta}\,\nabla f(s_1^\eta))_\eta \text{is uniformly bounded in } L^2(Q_T) \tag{4.3}$$
$$(\sqrt{M_i(s_i^\eta)}\,\nabla p_i^\eta)_\eta \text{is uniformly bounded in } L^2(Q_T) \tag{4.4}$$
$$(\beta(s_1^\eta))_\eta \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega)) \tag{4.5}$$
$$(\phi\partial_t(\rho_1(p_1^\eta)s_1^\eta))_\eta \text{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_1}(\Omega)') \tag{4.6}$$
$$(\phi\partial_t(s_2^\eta))_\eta \text{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_1}(\Omega)') \tag{4.7}$$

Démonstration. The maximum principle (4.1) is conserved through the limit process.
For the next four estimates, consider the $L^2(\Omega)$ scalar product of (1.27) by $g_1(p_1^\eta) = \int_0^{p_1^\eta} \frac{1}{\rho_1(\xi)}\,d\xi$ and (1.28) by p_2^η and adding them after denoting by $\mathcal{H}_1(p_1^\eta) = \rho_1(p_1^\eta)g_1(p_1^\eta) - p_1^\eta$, then we have

$$\frac{d}{dt}\int_\Omega \phi\left(s_1^\eta\mathcal{H}_1(p_1^\eta) + \int_0^{s_1^\eta} f(\xi)\,d\xi\right)dx + \eta\int_\Omega |\nabla f(s_1^\eta)|^2\,dx + \int_\Omega \mathbf{K}M_1(s_1^\eta)\nabla p_1^\eta\cdot\nabla p_1^\eta\,dx$$
$$+\int_\Omega \mathbf{K}M_2(s_2^\eta)\nabla p_2^\eta\cdot\nabla p_2^\eta\,dx - \int_\Omega \mathbf{K}M_1(s_1^\eta)\rho_1(p_1^\eta)\mathbf{g}\cdot\nabla p_1^\eta\,dx - \int_\Omega \mathbf{K}M_2(s_2^\eta)\rho_2\mathbf{g}\cdot\nabla p_2^\eta\,dx$$
$$+\int_\Omega \rho_1(p_1^\eta)s_1^\eta f_P g_1(p_1^\eta)\,dx + \int_\Omega s_2^\eta f_P p_2^\eta\,dx = \int_\Omega f_I p_2^\eta\,dx.$$

Using the assumptions (H1)–(H6), Cauchy Schwartz inequality, the functions g_1 is sublinear,

$$\frac{d}{dt}\int_\Omega \phi\left(s_1^\eta\mathcal{H}_1(p_1^\eta) + \int_0^{s_1^\eta} f(\xi)\,d\xi\right)dx + \int_\Omega \mathbf{K}M_1(s_1^\eta)\nabla p_1^\eta\cdot\nabla p_1^\eta\,dx + \eta\int_\Omega |\nabla f(s_1^\eta)|^2\,dx$$
$$+\int_\Omega \mathbf{K}M_2(s_2^\eta)\nabla p_2^\eta\cdot\nabla p_2^\eta\,dx - \int_\Omega \mathbf{K}M_1(s_1^\eta)\rho_1(p_1^\eta)\mathbf{g}\cdot\nabla p_1^\eta\,dx - \int_\Omega \mathbf{K}M_2(s_2^\eta)\rho_2\mathbf{g}\cdot\nabla p_2^\eta\,dx$$
$$= -\int_\Omega \rho_1(p_1^\eta)s_1^\eta f_P g_1(p_1^\eta)\,dx - \int_\Omega s_2^\eta f_P p_2^\eta\,dx + \int_\Omega f_I p_2^\eta\,dx$$
$$\le \frac{\rho_M}{\rho_m}(\| f_P \|_{L^2(\Omega)} + \| f_I \|_{L^2(\Omega)})(\| p_1^\eta \|_{L^2(\Omega)} + \| p_2^\eta \|_{L^2(\Omega)})$$
$$\le \frac{2\rho_M}{\rho_m}(\| f_P \|_{L^2(\Omega)} + \| f_I \|_{L^2(\Omega)})(\| p^\eta \|_{L^2(\Omega)} + \| \overline{p}(s_1^\eta) \|_{L^2(\Omega)} + \| \tilde{p}(s_1^\eta) \|_{L^2(\Omega)})$$
$$\le C(\| f_P \|_{L^2(\Omega)} + \| f_I \|_{L^2(\Omega)})(\| \nabla p^\eta \|_{L^2(\Omega)} + \| \overline{p}(s_1^\eta) \|_{L^2(\Omega)} + \| \tilde{p}(s_1^\eta) \|_{L^2(\Omega)}).$$

We have

$$\nabla p^\eta = \nabla p_2^\eta + \frac{M_1(s_1^\eta)}{M(s_1^\eta)}\nabla f(s_1^\eta) = \nabla p_1^\eta - \frac{M_2(s_2^\eta)}{M(s_1^\eta)}\nabla f(s_1^\eta), \tag{4.8}$$

then,

$$\int_{Q_T} M(s_1^\eta)|\nabla p^\eta|^2\,dxdt + \int_{Q_T} \frac{M_1(s_1^\eta)M_2(s_2^\eta)}{M(s_1^\eta)}|\nabla f(s_1^\eta)|^2\,dxdt$$
$$= \int_{Q_T} M_1(s_1^\eta)\nabla p_1^\eta\cdot\nabla p_1^\eta\,dxdt + \int_{Q_T} M_2(s_2^\eta)\nabla p_2^\eta\cdot\nabla p_2^\eta\,dxdt, \quad (4.9)$$

using again the assumptions (H1)–(H6), Cauchy Schwartz inequality, young inequality, the functions $\mathcal{H}_1$ is non negative and integrate the above inequality over $(0,T)$, we get

$$\int_{Q_T} M(s_1^\eta)|\nabla p^\eta|^2\,dxdt + \int_{Q_T} \frac{M_1(s_1^\eta)M_2(s_2^\eta)}{M(s_1^\eta)}|\nabla f(s_1^\eta)|^2\,dxdt$$
$$+ \int_{Q_T} M_1(s_1^\eta)\nabla p_1^\eta\cdot\nabla p_1^\eta\,dxdt + \int_{Q_T} M_2(s_2^\eta)\nabla p_2^\eta\cdot\nabla p_2^\eta\,dxdt$$
$$+\,\eta\int_{Q_T}\nabla f(s_1^\eta)\cdot\nabla f(s_1^\eta)\,dxdt \le C. \quad (4.10)$$

and this with the assumption (H3) ensure the estimate (4.2), the estimates (4.3)–(4.4) are established due to (4.10) . For the estimate (4.5) we have

$$\int_{Q_T} \frac{M_1(s_1^\eta)M_2(s_2^\eta)}{M(s_1^\eta)}|\nabla f(s_1^\eta)|^2\,dxdt \le C$$

using the assumption (H3),

$$\int_{Q_T} M_1(s_1^\eta)M_2(s_2^\eta)|\nabla f(s_1^\eta)|^2\,dxdt \le C,$$

which indicates that

$$\int_{Q_T} |\nabla\Gamma(s_1^\eta)|^2\,dxdt \le C, \quad (4.11)$$

where

$$\Gamma'(s_1) = \sqrt{M_1(s_1)M_2(s_2)}f'(s_1),$$

and this leads to the desired estimate (4.5).

$$\int_{Q_T} |\nabla\beta(s_1^\eta)|^2\,dxdt = \int_{Q_T} \frac{M_1^2(s_1^\eta)M_2^2(s_2^\eta)}{M^2(s_1^\eta)}|\nabla f(s_1^\eta)|^2\,dxdt$$
$$\le \frac{1}{4}\int_{Q_T} M_1(s_1^\eta)M_2(s_2^\eta)|\nabla f(s_1^\eta)|^2\,dxdt \le C.$$

For all $\varphi, \xi \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$, we have

$$\langle \phi\partial_t(\rho_1(p_1^\eta)s_1^\eta), \varphi\rangle + \int_{Q_T} \mathbf{K}M_1(s_1^\eta)\rho_1(p_1^\eta)\nabla p_1^\eta \cdot \nabla\varphi \, dxdt \\ - \int_{Q_T} \mathbf{K}M_1(s_1^\eta)\rho_1^2(p_1^\eta)\mathbf{g} \cdot \nabla\varphi \, dxdt + \eta \int_{Q_T} \rho_1(p_1^\eta)\nabla(p_1^\eta - p_2^\eta) \cdot \nabla\varphi \, dxdt \\ + \int_{Q_T} \rho_1(p_1^\eta)s_1^\eta f_P \varphi \, dxdt = 0 \quad (4.12)$$

$$\langle \phi\partial_t(s_2^\eta), \xi\rangle + \int_{Q_T} \mathbf{K}M_2(s_2^\eta)\nabla p_2^\eta \cdot \nabla\xi \, dxdt \\ - \int_{Q_T} \mathbf{K}M_2(s_2^\eta)\rho_2\mathbf{g} \cdot \nabla\xi \, dxdt - \eta \int_{Q_T} \nabla(p_1^\eta - p_2^\eta) \cdot \nabla\xi \, dxdt \\ + \int_{Q_T} s_2^\eta f_P \xi \, dxdt = \int_{Q_T} f_I \xi \, dxdt \quad (4.13)$$

where the bracket $\langle \cdot, \cdot \rangle$ represents the duality product between $L^2(0,T;(H^1_{\Gamma_1}(\Omega))')$ and $L^2(0,T;H^1_{\Gamma_1}(\Omega))$. Using (4.8), one gets

$$|\langle \phi\partial_t(\rho_1(p_1^\eta)s_1^\eta), \varphi\rangle| \le \left|\eta \int_{Q_T} \rho_1(p_1^\eta)\nabla f(s_1^\eta) \cdot \nabla\varphi \, dxdt\right| \\ + \left|\int_{Q_T} \mathbf{K}\rho_1(p_1^\eta)(M_1(s_1^\eta)\nabla p^\eta + \nabla\beta(s_1^\eta)) \cdot \nabla\varphi \, dxdt\right| \\ + \left|\int_{Q_T} \mathbf{K}\rho_1^2(p_1^\eta)M_1(s_1^\eta)\mathbf{g} \cdot \nabla\varphi \, dxdt\right| + \left|\int_{Q_T} \rho_1(p_1^\eta)s_1^\eta f_P \varphi \, dxdt\right|, \quad (4.14)$$

The functions ρ_1 and M_1 are bounded, with Cauchy Schwarz inequality imply,

$$|\langle \phi\partial_t(\rho_1(p_1^\eta)s_1^\eta), \varphi\rangle| \le C\Big(1 + \|\nabla p^\eta\|_{L^2(Q_T)} + \|\nabla\beta(s^\eta)\|_{L^2(Q_T)} \\ + \|\eta\nabla f(s^\eta)\|_{L^2(Q_T)}\Big)\|\nabla\varphi\|_{L^2(Q_T)} \quad + \rho_M\|f_P\|_{L^2(Q_T)}\|\varphi\|_{L^2(Q_T)},$$

and from the estimates (4.2)–(4.5), we deduce

$$|\langle \phi\partial_t(\rho_1(p_1^\eta)s_1^\eta), \varphi\rangle| \le C \parallel \varphi \parallel_{L^2(0,T;H^1_{\Gamma_1}(\Omega))},$$

which establish (4.6). In the same way we obtain the estimate (4.7). Indeed, for $\xi \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$ and from (2.2) we have

$$|\langle \phi\partial_t s_2^\eta, \xi\rangle| \le \left|\eta \int_{Q_T} \nabla f(s_1^\eta) \cdot \nabla\xi \, dxdt\right| + \left|\int_{Q_T} \mathbf{K}(M_2(s_2^\eta)\nabla p^\eta + \nabla\beta(s_1^\eta)) \cdot \nabla\xi \, dxdt\right| \\ + \left|\int_{Q_T} \mathbf{K}\rho_2 M_2(s_2^\eta)\mathbf{g} \cdot \nabla\xi \, dxdt\right| + \left|\int_{Q_T} s_2^\eta(f_P - f_I)\xi \, dxdt\right|, \\ (4.15)$$

and using the Cauchy Schwarz inequality as above, similarly we get

$$|\langle \phi\partial_t s_2^\eta, \xi\rangle| \le C\Big(1 + \|\nabla p^\eta\|_{L^2(Q_T)} + \|\nabla\beta(s^\eta)\|_{L^2(Q_T)} + \|\eta\nabla f(s^\eta)\|_{L^2(Q_T)}\Big)\|\nabla\xi\|_{L^2(Q_T)} + C \parallel f_P - f_I \parallel_{L^2(Q_T)} \|\xi\|_{L^2(Q_T)}.$$

and from the estimates (4.2)–(4.5), we deduce

$$|\langle \phi\partial_t s_2^\eta, \xi\rangle| \le C \parallel \xi \parallel_{L^2(0,T;H^1_{\Gamma_1}(\Omega))},$$

which establish (4.7), and close the proof of the lemma. □

From the previous two lemmas, we deduce the following convergences.
Lemma 2.8. *(Strong and weak convergences)*
Let $f_1, f_2 \in C_b^0(\mathbb{R})$ such that $f_2(0) = 0$, then up to a subsequence, the sequences $(s_i^\eta)_\eta$, $(p^\eta)_\eta$, $(p_i^\eta)_\eta$ verify the following convergence

$$\begin{aligned}
&s_1^\eta \longrightarrow s_1 && \textit{almost everywhere in } Q_T \textit{ and } \Sigma_T, && (4.16)\\
&s_1^\eta \longrightarrow s_1 && \textit{strongly in } L^2(Q_T) \textit{ and } L^2(\Sigma_T), && (4.17)\\
&0 \le s_i(t,x) \le 1 && \textit{almost everywhere in } (t,x)\in Q_T, && (4.18)\\
&\beta(s_1^\eta) \longrightarrow \beta(s_1) && \textit{weakly in } L^2(0,T;H^1(\Omega)), && (4.19)\\
&p^\eta \longrightarrow p && \textit{weakly in } L^2(0,T;H^1_{\Gamma_1}(\Omega)), && (4.20)\\
&\phi\partial_t s_2^\eta \longrightarrow \phi\partial_t s_2 && \textit{weakly in } L^2(0,T;(H^1_{\Gamma_1}(\Omega))'), && (4.21)\\
&f_1(p_1^\eta)f_2(s_1^\eta) \longrightarrow f_1(p_1)f_2(s_1) && \textit{almost everywhere in } (t,x)\in Q_T, && (4.22)\\
&\phi\partial_t(\rho_1(p_1^\eta)s_1^\eta) \longrightarrow \phi\partial_t(\rho_1(p_1)s_1) && \textit{weakly in } L^2(0,T;(H^1_{\Gamma_1}(\Omega))'). && (4.23)
\end{aligned}$$

Démonstration. The relative compactness of the sequence $(s_1^\eta)_\eta$ in the degenerate case is obtained in the same manner as in the incompressible flows. For that, from (H4) the function β^{-1} is an Holder function of order θ, with $0 < \theta \le 1$, and the sequence $(s_1^\eta)_\eta$ satisfy (4.1), (4.5) and (4.7), then up to subsequence we have

$$s_1^\eta \longrightarrow s_1 \text{ strongly in } L^2(Q_T), \tag{4.24}$$

$$s_1^\eta \longrightarrow s_1 \text{ strongly in } L^2(\Sigma_T), \tag{4.25}$$

this compactness result is proved in ([22]) in the case where the porosity is constant, a straightforward modification of the Lemma 3.1 in [44] shows that the compactness lemma remains valid under the assumption $(H1)$.

Lemma 2.9. *(Compactness result for degenerate case) For every M, the set*

$$E_M = \{(s \in L^2(Q_T), 0 \le s(t,x) \le 1 \text{ a.e. in } Q_T \text{ such that}$$
$$\|\beta(s)\|_{L^2(0,T;H^1(\Omega))} \le M,\ \|\phi \partial_t s\|_{L^2(0,T;(H^1_{\Gamma_1}(\Omega))')} \le M\}$$

is relatively compact in $L^2(Q_T)$, and $\gamma(E_M)$ is relatively compact in $L^2(\Sigma_T)$, (γ denotes the trace on Σ_T operator).

Démonstration. The proof is based on the proof of the Lemma 3 in ([22], p. 140). For that, for $0 < \tau < 1$, and $1 < r < \infty$, let us denote

$$W^{\tau,r}(\Omega) = \{w \in L^r(\Omega); \int_\Omega \int_\Omega \frac{|w(x) - w(y)|^r}{|x-y|^{d+\tau r}}\, dxdy < +\infty\}$$

equipped with the norm

$$\|w\|_{W^{\tau,r}(\Omega)} = \left(\|w\|^r_{L^r(\Omega)} + \int_\Omega \int_\Omega \frac{|w(x) - w(y)|^r}{|x-y|^{d+\tau r}}\, dxdy \right)^{\frac{1}{r}},$$

recall d denote the space dimension. Denote $v = \beta(s)$, then $v \in L^2(0,T;H^1(\Omega))$ and $s = \beta^{-1}(v)$ is an Hölder function. Using the continuity of the injection of $H^1(\Omega)$ into $W^{\tau,2}(\Omega)$ for any $0 < \tau < 1$, we have $v \in L^2(0,T;W^{\tau,2}(\Omega))$. So, $v(t,\cdot) \in W^{\tau,2}(\Omega)$ a.e. in (0,T), which implies that $s(t,\cdot) = \beta^{-1}(v(t,\cdot)) \in W^{\tau\theta,2/\theta}(\Omega)$ and

$$\|s\|_{W^{\theta\tau,r/\theta}(\Omega)} \le c\|v\|^\theta_{W^{\tau,r}(\Omega)}.$$

Integrating the above inequality over $(0,T)$,

$$\|s\|_{L^{2/\theta}(0,T;W^{\theta\tau,2/\theta}(\Omega))} \le c\|v\|^\theta_{L^2(0,T;H^1(\Omega))}.$$

Furthermore the porosity function ϕ belongs to $W^{1,\infty}(\Omega)$, it follows that

$$\|\phi s\|_{L^{2/\theta}(0,T;W^{\theta\tau,2/\theta}(\Omega))} \le C.$$

As Ω is bounded and regular, we have, for $\tau' < \theta\tau$,

$$W^{\theta\tau,2/\theta}(\Omega) \subset W^{\tau',2/\theta}(\Omega) \subset (H^1_{\Gamma_1}(\Omega))'$$

with compact injection from $W^{\theta\tau,2/\theta}(\Omega)$ into $W^{\tau',2/\theta}(\Omega)$. Now, from a standard compactness argument (see [63]), we get

$$E_M \text{ is relatively compact in } L^{2/\theta}(0,T;W^{\tau',2/\theta}(\Omega)) \subset L^2(0,T;L^2(\Omega)).$$

Secondly, the trace operator γ maps continuously $W^{\tau',2/\theta}(\Omega)$ into $W^{\tau'-\theta/2,2/\theta}(\Gamma)$ as soon as $\tau' > \theta/2$. Choosing for example $\tau' = \frac{3\theta}{4}$, we deduce the relative compactness of $\gamma(E_M)$ into $L^2(\Sigma_T)$. This closes the proof of lemma 2.9. $\square$

Consequently the maximum principle (4.18) is conserved through a limit process. The weak convergences (4.19)-(4.20) on the capillary term β and the global pressure p are a direct consequences of (4.5) and (4.2). The convergence (4.21) is a consequence of the estimate (4.7), and the identification of the limit follows from the above convergences.

Now, for the convergence of the last two terms, define $\gamma(s_1) = s_1\alpha^3(s_1)$, and let us prove

$$\rho_1(p_1^\eta)\gamma(s_1^\eta))_\eta \longrightarrow \gamma(s_1)\rho_1(p_1),\ \text{for all } (t,x) \text{ in } (0,T)\times\Omega \text{ almost everywhere.} \tag{4.26}$$

The sequence $(\rho_1(p_1^\eta)\gamma(s_1^\eta))_\eta$ is uniformly bounded in $L^2(0,T;H^1_{\Gamma_1}(\Omega))$: since

$$\begin{aligned}\nabla(\rho_1(p_1^\eta)\gamma(s_1^\eta)) &= \gamma(s_1^\eta)\nabla\rho_1(p_1^\eta) + \Big(3s_1^\eta\alpha(s_1^\eta)\alpha'(s_1^\eta) + \alpha^2(s_1^\eta)\Big)\nabla\beta(s^\eta) \\ &= s_1^\eta\alpha^2(s_1^\eta)\rho_1'(p_1^\eta)\frac{\sqrt{M_1(s_1^\eta)}M_2(s_2^\eta)}{M(s_1^\eta)}f'(s_1^\eta)\Big(\sqrt{M_1(s_1^\eta)}\nabla p_1^\eta\Big) \\ &\quad + \Big(3s_1^\eta\alpha(s_1^\eta)\alpha'(s_1^\eta) + \alpha^2(s_1^\eta)\Big)\nabla\beta(s_1^\eta),\end{aligned} \tag{4.27}$$

the uniform bound becomes a consequence of (4.4)-(4.5).
The sequence $(\phi\partial_t(\rho(p_1^\eta)\gamma(s_1^\eta))_\eta$ is uniformly bounded in $\Big(L^2(0,T;H^1_{\Gamma_w}(\Omega))\cap L^\infty(Q_T)\Big)'$. In fact,

$$\langle\phi\partial_t(\rho_1(p_1^\eta)\gamma(s_1^\eta),\varphi\rangle = \langle\phi\partial_t(\rho_1(p_1^\eta)s_1^\eta),\alpha^3(s_1^\eta)\varphi\rangle + \langle\phi\partial_t s_1^\eta, 3\rho_1(p_1^\eta)s_1^\eta\alpha^2(s_1^\eta)\alpha'(s_1^\eta)\varphi\rangle \tag{4.28}$$

for all $\varphi\in L^2(0,T;H^1_{\Gamma_w}(\Omega))\cap L^\infty(Q_T)$. Therefore, the evolution term is uniformly bounded due to (4.6)-(4.7).
Then up to a subsequence, we have

$$\rho_1(p_1^\eta)\gamma(s_1^\eta) \longrightarrow l \text{ strongly in } L^1(0,T;H^{\frac{1}{2}}(\Omega)).$$

This result is a direct consequence of a Simon's Lemma [62] when the porosity is constant, and under the assumption (H1), we use Lemma 2.9. We have also,

$$\rho_1(p_1^\eta)\gamma(s_1^\eta) \longrightarrow l \text{ strongly in } L^1(0,T;L^2(\Sigma_T)). \tag{4.29}$$

From the Lebesgue theorem, we have

$$\rho_1(p_1^\eta)\gamma(s_1^\eta) \longrightarrow l \text{ strongly in } L^q(Q_T), \text{ for all } 1 \le q < \infty. \tag{4.30}$$

Now, due to the monotonicity of the function ρ_1, we have

$$\int_{Q_T} \Big(\gamma(s_1^\eta)\rho_1(p_1^\eta) - \gamma(s_1^\eta)\rho(v)\Big)(p_1^\eta - v)dtdx \ge 0, \qquad \forall v \in L^2(Q_T),$$

Note that, from the relation between p^η and p_1^η and by the help of (4.17) and (4.20) we obtain

$$p_1^\eta \longrightarrow p_1 \text{ weakly in } L^2(Q_T), \tag{4.31}$$

this with the convergence results (4.30), (4.24) leads to,

$$\int_{Q_T} (l - \gamma(s)\rho_1(v))(p_1 - v)\, dtdx \ge 0, \qquad \forall v \in L^2(Q_T).$$

Finally, choose $v = p_1 - \delta w$ with $\delta \in]0, 1]$, $w \in L^2(Q_T)$, then

$$\int_{Q_T} (l - \gamma(s)\rho_1(p_1 - \delta w))w\, dtdx \ge 0,$$

letting δ goes to zero, we establish (4.26)

$$l = \gamma(s_1)\rho_1(p_1), \text{ for all } (t, x) \text{ in } (0, T) \times \Omega \text{ almost everywhere.} \tag{4.32}$$

To conclude the almost everywhere convergence (4.22), consider $s_1^\eta \to s_1 = 0$ almost everywhere then $f_1(p_1^\eta)f_2(s_1^\eta) \to 0 = f_1(p_1)f_2(s_1)$ almost everywhere, since $f_2(0) = 0$ and f_1 is bounded. Next, when $s_1^\eta \to s_1 \neq 0$, in light of (4.26) and due to the assumption that α vanishes only at zero, we have $f_1(p_1^\eta) \to f_1(p_1)$ almost everywhere, and $f_1(p_1^\eta)f_2(s_1^\eta) \to f_1(p_1)f_2(s_1)$ by the continuity of the functions f_1 and f_2.
Finally (4.23) is a consequence of (4.6), and the identification of the limit is due to (4.22). □

Limit process.
In order to achieve the proof of Theorem 2.1, it remains to pass to the limit as η goes to zero in the formulations (1.30)–(1.31), for all smooth test functions

$\varphi \in C^1([0,T]; H^1_{\Gamma_1}(\Omega))$ such that $\varphi(T) = 0$

$$
\begin{aligned}
-\int_{Q_T} \phi\rho_1(p_1^\eta)s_1^\eta \partial_t\varphi \, dxdt + \int_{Q_T} \mathbf{K}M_1(s_1^\eta)\rho_1(p_1^\eta)\nabla p_1^\eta \cdot \nabla\varphi \, dxdt \\
- \int_{Q_T} \mathbf{K}M_1(s_1^\eta)\rho_1^2(p_1^\eta)\mathbf{g} \cdot \nabla\varphi \, dxdt + \int_{Q_T} \rho_1(p_1^\eta)s_1^\eta f_P\varphi \, dxdt \\
+ \eta \int_\Omega \rho_1(p_1^\eta)\nabla(p_1^\eta - p_2^\eta)\nabla\varphi \, dxdt = \int_\Omega \phi\rho_1(p_1^0)s_1^0\varphi(0,x) \, dxdt.
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
-\int_{Q_T} \phi s_2^\eta \partial_t\varphi \, dxdt + \int_{Q_T} \mathbf{K}M_2(s_2^\eta)\nabla p_2^\eta \cdot \nabla\varphi \, dxdt \\
- \int_{Q_T} \mathbf{K}M_2(s_2^\eta)\rho_2\mathbf{g} \cdot \nabla\varphi \, dxdt + \int_{Q_T} s_2^\eta f_P\varphi \, dxdt \\
- \eta \int_\Omega \nabla(p_1^\eta - p_2^\eta)\nabla\varphi \, dxdt = \int_{Q_T} f_I\varphi \, dxdt + \int_\Omega \phi s_2^0\varphi(0,x) \, dxdt.
\end{aligned} \tag{4.34}
$$

The first and third term in (4.33) converge due to the strong convergence of $\rho_i(p_i^\eta)s_i^\eta$ to $\rho_i(p_i)s_i$ in $L^2(Q_T)$.
The second term can be written as,

$$
\begin{aligned}
\int_{Q_T} \mathbf{K}M_1(s_1^\eta)\rho_1(p_1^\eta)\nabla p_1^\eta \cdot \nabla\varphi \, dxdt = \int_{Q_T} \mathbf{K}M_1(s_1^\eta)\rho_1(p_1^\eta)\nabla p^\eta \cdot \nabla\varphi \, dxdt \\
+ \int_{Q_T} \mathbf{K}\rho_1(p_1^\eta)\frac{\sqrt{M_1(s_1^\eta)M_2(s_1^\eta)}}{M(s_1^\eta)}\nabla\Gamma(s_1^\eta) \cdot \nabla\varphi \, dxdt
\end{aligned} \tag{4.35}
$$

The first term on the right hand side of the equation (4.35) converges arguing in two steps. Firstly, let the functions f_1, f_2 of the previous *lemma* play respectively the role of ρ_1, M_1 and by Lebesgue theorem we get,

$$\rho_1(p_1^\eta)M_1(s_1^\eta)\nabla\varphi \longrightarrow \rho_1(p_1)M_1(s_1)\nabla\varphi \text{ strongly in } (L^2(Q_T))^d.$$

Secondly, as a consequence of (4.2), we have

$$\nabla p^\eta \longrightarrow \nabla p \quad \text{weakly in } (L^2(Q_T))^d,$$

Now, this weak convergence combined to the above strong convergence validate the convergence of the first term in (4.35). Similarly, the last term on the right hand side of the equation (4.35) converges arguing in two steps. Firstly, let the functions f_1, f_2 play respectively the role of $\rho_1, \frac{\sqrt{M_1M_2}}{M}$ and by Lebesgue theorem we get,

$$\rho_1(p_1^\eta)\frac{\sqrt{M_1(s_1^\eta)M_2(s_1^\eta)}}{M(s_1^\eta)}\nabla\varphi \longrightarrow \rho_1(p_1)\frac{\sqrt{M_1(s_1)M_2(s_2)}}{M(s_1)}\nabla\varphi \text{ strongly in } (L^2(Q_T))^d.$$

Now, as a consequence of (4.11),

$$\nabla\Gamma(s_1^\eta) \longrightarrow \nabla\Gamma(s_1) \quad \text{weakly in } (L^2(Q_T))^d,$$

Now, this weak convergence combined to the above strong convergence validate the convergence of the last term in (4.35). and this achieves the passage to the limit on the second term of (4.33).
The passage to the limit in the third term attain by Lebesgue theorem and (4.22), where f_1 plays the role of ρ_1^2 and f_2 plays the role of M_1.

The fifth term can be written as,

$$\eta\int_\Omega \rho_i(p_i^\eta)\nabla(p_1^\eta - p_2^\eta)\nabla\varphi\,dxdt = \sqrt{\eta}\int_\Omega \rho_i(p_i^\eta)(\sqrt{\eta}\nabla f(s_1^\eta))\cdot\nabla\varphi\,dxdt, \qquad (4.36)$$

the Cauchy-Schwartz inequality and the uniform estimate (4.3) ensure the convergence of this term to zero.
Similarly, we can pass the limit to (4.34). The weak formulations (1.16) and (1.17) are then established.
The main theorem 2.1 is then established.

CHAPITRE 3

Two compressible immiscible flow in porous media

Abstract. In this paper, we consider a model of flow of two compressible and immiscible phases in a three dimensional porous media. The equations are obtained by the conservation of the mass of each phase. This model is treated in its general form with the whole nonlinear terms. We establish an existence result for this model.

1 Introduction, Assumptions and Main Results

Many authors studied flows in porous media. The study of the miscible flow models has been investigated in ([10], [11], [40]) and recently in ([17], [18], [19]). The immiscible and incompressible flows have been treated by many authors ([10], [9], [22], [39], [29], [41], [42]). For two immiscible compressible flows, we refer to [44], [47], and recently [45] and [15].
The immiscible flow models developed by [44], [45], [47] use the feature of global pressure even if the density of each phase depends on its own pressure, then the context was to assume small capillary pressure so that the densities are assumed to depend on the global pressure, recently and under that context C. Galusinski, M. Saad [45] obtained an existence result of solutions.
In this paper, we consider the two compressible immiscible flows model studied in [45], with the difference that we will not use the feature of global pressure in the sense that it enables us to write all models with one pressure variable and one or several saturations with assumption concerns the dependence of densities on a global pressure. The model is treated in its general form under the physical assumption that the density of each phase depends on its own pressure. The

mathematical analysis of this model is based on new energy estimates on the pressures. The main idea consists to derive from degenerate estimates on pressure of each phase, which not allowed straight bound on pressures, an estimate on global pressure and degenerate capillary term. An appropriate compactness lemma is shown with the help of the feature of global pressure to pass from non-degenerate case to the degenerate case.
We give below the basic model written in variables pressures and saturations.
The equations describing the immiscible displacement of two compressible fluids are given by the following mass conservation of each phase $(i = 1, 2)$:

$$\phi(x)\partial_t(\rho_i(p_i)s_i)(t,x) + \operatorname{div}(\rho_i(p_i)\mathbf{V}_i)(t,x) + \rho_i(p_i)s_i f_P(t,x) = \rho_i(p_i)s_i^I f_I(t,x), \tag{1.1}$$

where ϕ is the porosity of the medium, ρ_i and s_i are respectively the density and the saturation of the i^{th} fluid. The velocity of each fluid $\mathbf{V}_i$ is given by the Darcy law :

$$\mathbf{V}_i(t,x) = -\mathbf{K}(x)\frac{k_i(s_i(t,x))}{\mu_i}\Big(\nabla p_i(t,x) - \rho_i(p_i)\mathbf{g}\Big), \qquad i = 1, 2. \tag{1.2}$$

where $\mathbf{K}(x)$ is the permeability tensor of the porous medium at point x to the fluid under consideration, k_i the relative permeability of the i^{th} phase, μ_i the constant i-phase's viscosity, p_i the i-phase's pressure and $\mathbf{g}$ is the gravity term. Here the functions f_I and f_P are respectively the injection and production terms. Note that in equation (1.1) the injection term is multiplied by a known saturation s_i^I corresponding to the known injected fluid, whereas the production term is multiplied by the unknown saturation s_i corresponding to the produced fluid. By definition of saturations, one has

$$s_1(t,x) + s_2(t,x) = 1. \tag{1.3}$$

The curvature of the contact surface between the two fluids links the jump of pressure of the two phases to the saturation by the capillary pressure law in order to close the system (1.1)-(1.3),

$$f(s_1(t,x)) = p_1(t,x) - p_2(t,x). \tag{1.4}$$

With the arbitrary choice of (1.4) (the jump of pressure is a function of s_1), the application $s_1 \mapsto f(s_1)$ is non-decreasing, $(\frac{df}{ds_1}(s_1) > 0,$ for all $s_1 \in [0,1])$, and usually $f(s_1 = 1) = 0$, in the case of two phases, when the wetting fluid is at its maximum saturation. In order to know which of the fluids is the wetting one, one has to look at the meniscus separating the two fluids in a capillary tube, the concavity of the meniscus is oriented towards the non wetting fluid. For example, air is the non wetting phase in water air displacement. In this study we consider the index $i = 1$ represents the wetting fluid, and for this choice capillary pressure vanishes when $s_1 = 1$. This point is crucial in determining the spaces that the saturation of each phase belongs. We take the capillary pressure function f as considered in [22], defined on $[0, 1]$, increasing and $f(1) = 0$.

In section 4 we will use the feature of global pressure. For that let us denote,

$$\begin{aligned} M_i(s_i) &= k_i(s_i)/\mu_i && i - \text{phase's mobility,} \\ M(s_1) &= M_1(s_1) + M_2(1 - s_1) && \text{the total mobility,} \\ \mathbf{V} &= \mathbf{V}_1 + \mathbf{V}_2 && \text{the total velocity.} \end{aligned}$$

As in [22], [64] and [45] we can express the total velocity in terms of p_2 and $f(s_1)$. We have

$$\mathbf{V} = -\mathbf{K}M(s_1)\left(\nabla p_2 + \frac{M_1(s_1)}{M(s_1)}\nabla f(s_1)\right) + \mathbf{K}(M_1(s_1)\rho_1(p_1) + M_2(s_2)\rho_2(p_2))\mathbf{g};$$

and defining the functions $\tilde{p}(s_1)$, $\bar{p}(s_1)$ such that

$$\tilde{p}'(s_1) = \frac{M_1(s_1)}{M(s_1)}f'(s_1), \quad \text{and } \bar{p}'(s_1) = -\frac{M_2(s_2)}{M(s_1)}f'(s_1), \tag{1.5}$$

the global pressure is then defined as in [22]

$$p = p_2 + \tilde{p}(s_1) = p_1 + \bar{p}(s_1) \tag{1.6}$$

Thus, the total velocity can be expressed as

$$\mathbf{V} = -\mathbf{K}M(s_1)\nabla p + \mathbf{K}(M_1(s_1)\rho_1(p_1) + M_2(s_2)\rho_2(p_2))\mathbf{g},$$

and each fluid velocity

$$\mathbf{V}_i = -\mathbf{K}M_i(s_i)\nabla p - \mathbf{K}\alpha(s_1)\nabla s_i + \mathbf{K}M_i(s_i)\rho_i(p_i)\mathbf{g}.$$

where $\alpha(s_1) = \frac{M_1(s_1)M_2(s_2)}{M(s_1)}\frac{df}{ds}(s_1) \geq 0$.
Define

$$\beta(s) = \int_0^s \alpha(\xi)d\xi. \tag{1.7}$$

In this paper we do not use this concept of writing the total and each velocity in terms of the global pressure and one saturation, but just to show the source of definitions of some functions.

We detail the physical context by introducing the boundary conditions, the initial conditions and some assumptions on the data of the problem.
Let $T > 0$, fixed and let Ω be a bounded set of $\mathbb{R}^d$ ($d \geq 1$). We set $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$. To the system (1.1)-(1.3)-(1.4) ($i = 1,2$), we add the following mixed boundary conditions and initial conditions. We consider the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_{imp}$, where Γ_1 denotes the injection boundary of the first phase and Γ_{imp} the impervious one.

$$\begin{cases} p_1(t,x) = 0, \quad p_2(t,x) = 0 & \text{on } \Gamma_1 \\ \mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0 & \text{on } \Gamma_{imp} \end{cases} \tag{1.8}$$

where $\mathbf{n}$ is the outward normal to the boundary Γ_{imp}.
The initial conditions are defined on pressures

$$\begin{cases} p_1(0,x) = p_1^0(x) \text{ in } \Omega \\ p_2(0,x) = p_2^0(x) \text{ in } \Omega. \end{cases} \tag{1.9}$$

Next we are going to introduce some physically relevant assumptions on the coefficients of the system.

($H1$) The porosity $\phi \in W^{1,\infty}(\Omega)$ and there is two positive constants ϕ_0 and ϕ_1 such that $\phi_0 \leq \phi(x) \leq \phi_1$ almost everywhere $x \in \Omega$.

($H2$) The tensor $\mathbf{K}$ belongs to $(W^{1,\infty}(\Omega))^{d\times d}$. Moreover, there exist two positive constants k_0 and k_∞ such that

$$\|\mathbf{K}\|_{(L^\infty(\Omega))^{d\times d}} \leq k_\infty \text{ and } (\mathbf{K}(x)\xi, \xi) \geq k_0|\xi|^2, \text{ for all } \xi \in \mathbb{R}^d, \text{ a.e. } x \in \Omega.$$

($H3$) The functions M_1 and M_2 belong to $\mathcal{C}^0([0,1];\mathbb{R}^+)$, $M_1(s_1 = 0) = 0$ and $M_2(s_2 = 0) = 0$. In addition, there is a positive constant m_0, such that, for all $s_1 \in [0,1]$,

$$M_1(s_1) + M_2(s_2) \geq m_0.$$

($H4$) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \geq 0$ almost everywhere $(t,x) \in Q_T$, $s_i^I(t,x) \geq 0$ ($i = 1,2$) and $s_1^I(t,x) + s_2^I(t,x) = 1$ almost everywhere in $(t,x) \in Q_T$.

($H5$) The densities ρ_i ($i = 1,2$) are $\mathcal{C}^2(\mathbb{R})$, increasing and there exist two positive constants ρ_m *and* ρ_M such that $0 < \rho_m \leq \rho_i(p_i) \leq \rho_M$, for $i = 1,2$.

($H6$) The capillary pressure function $f \in \mathcal{C}^1([0,1];\mathbb{R}^-)$ and $0 < \underline{f} \leq \frac{df}{ds}$.

($H7$) The function $\alpha \in \mathcal{C}^1([0,1];\mathbb{R}^+)$ satisfies $\alpha(s) > 0$ for $0 < s < 1$, and $\alpha(0) = \alpha(1) = 0$.

We assume that β^{-1}, inverses of $\beta(s) := \int_0^s \alpha(z)dz$, is an Hölder function of order θ, with $0 < \theta \leq 1$, on $[0, \beta(1)]$. Which means there exists a positive c such that for all $s_1, s_2 \in [0, \beta(1)]$, one has $|\beta^{-1}(s_1) - \beta^{-1}(s_1)| \leq c|s_1 - s_2|^\theta$.

The assumptions (H1)–(H7) are classical for porous media.

The main existence result of this paper is given below, for that let us define the following Sobolev space

$$H^1_{\Gamma_1}(\Omega) = \{\, u \in H^1(\Omega)\, ;\, u = 0 \text{ on } \Gamma_1 \,\},$$

this is an Hilbert space when equipped with the norm $\|u\|_{H^1_{\Gamma_1}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.
Let us state the main results of this paper.

Theorem 3.1. *Let (H1)-(H7) hold. Let $(p_1^0,\, p_2^0)$ belongs to $L^2(\Omega) \times L^2(\Omega)$. Then there exists (p_1, p_2) satisfying*

$$p_i \in L^2(0,T; H^1_{\Gamma_1}(\Omega)),\;\; \phi\partial_t(\rho_i(p_i)s_i) \in L^2(0,T; (H^1_{\Gamma_1}(\Omega))'),\;\; i = 1, 2, \tag{1.10}$$
$$0 \leq s_i(t,x) \leq 1 \;\; a.e \text{ in } Q_T,\;\; i = 1, 2,\;\; \beta(s_1) \in L^2(0,T; H^1(\Omega)) \tag{1.11}$$

such that for all $\varphi,\, \xi \in C^1(0,T; H^1_{\Gamma_1}(\Omega))$ with $\varphi(T) = \xi(T) = 0$,

$$\begin{aligned}
&- \int_{Q_T} \phi\rho_1(p_1)s_1\partial_t\varphi\, dxdt - \int_\Omega \phi(x)\rho_1(p_1^0(x))s_1^0(x)\varphi(0,x)\, dx \\
&+ \int_{Q_T} \mathbf{K}M_1(s_1)\rho_1(p_1)\nabla p_1 \cdot \nabla\varphi\, dxdt - \int_{Q_T} \mathbf{K}M_1(s_1)\rho_1^2(p_1)\mathbf{g} \cdot \nabla\varphi\, dxdt \\
&+ \int_{Q_T} \rho_1(p_1)s_1 f_P\varphi\, dxdt = \int_{Q_T} \rho_1(p_1)s_1^I f_I\,\varphi\, dxdt,
\end{aligned} \tag{1.12}$$

$$\begin{aligned}
&- \int_{Q_T} \phi\rho_2(p_2)s_2\partial_t\xi\, dxdt - \int_\Omega \phi(x)\rho_2(p_2^0(x))s_2^0(x)\xi(0,x)\, dx \\
&+ \int_{Q_T} \mathbf{K}M_2(s_2)\rho_2(p_2)\nabla p_2 \cdot \nabla\xi\, dxdt - \int_{Q_T} \mathbf{K}M_2(s_2)\rho_2^2(p_2)\mathbf{g} \cdot \nabla\xi\, dxdt \\
&+ \int_{Q_T} \rho_2(p_2)s_2 f_P\xi\, dxdt = \int_{Q_T} \rho_2(p_2)s_2^I f_I\,\xi\, dxdt,
\end{aligned} \tag{1.13}$$

and finally the initial conditions are satisfied in a weak sense as follows :

$$\text{For all } \psi \in H^1_{\Gamma_1}(\Omega) \text{ the function } t \longrightarrow \int_\Omega \phi\rho_i(p_i)s_i\psi\, dx \in \mathcal{C}^0([0,T]), \tag{1.14}$$

furthermore we have

$$\Big(\int_\Omega \phi\rho_i(p_i)s_i\psi\, dx\Big)(0) = \int_\Omega \phi\rho_i(p_i^0)s_i^0\psi\, dx. \tag{1.15}$$

As we can see, the above notion of weak solutions is very natural provided that we

explain the origin of the requirements (1.10)–(1.11). Obviously, they correspond to *a priori* estimates. Indeed, the equations (1.12)-(1.13) ensure that $s_i \geq 0$ ($i = 1, 2$) which is equivalent to $0 \leq s_i \leq 1$ (the proof is detailed in lemma 3.5. The key point is to obtain the estimates on ∇p and $\nabla \beta(s_1)$.
For that, define

$$g_i(p_i) := \int_0^{p_i} \frac{1}{\rho_i(\xi)}\, d\xi, \qquad i = 1, 2, \tag{1.16}$$

$$\mathcal{H}_i(p_i) := \rho_i(p_i) g_i(p_i) - p_i \qquad i = 1, 2, \tag{1.17}$$

then $\mathcal{H}_i'(p_i) = \rho_i'(p_i) g_i(p_i)$, $\mathcal{H}_i(0) = 0$, $\mathcal{H}_i(p_i) \geq 0$ for all p_i, and $\mathcal{H}_i$ is sublinear with respect to p_i. Multiplying (1.1) by $g_1(p_1)$ for $i = 1$ and (1.1) by $g_2(p_2)$ for $i = 2$ then integrate the equations with respect to x and adding them, we deduce at least formally,

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \phi \left(s_1 \mathcal{H}_1(p_1) + s_2 \mathcal{H}_2(p_2) + \int_0^{s_1} f(\xi)\, d\xi \right) dx \\
&+ \int_\Omega \mathbf{K} M_1(s_1) \nabla p_1 \cdot \nabla p_1\, dx + \int_\Omega \mathbf{K} M_2(s_2) \nabla p_2 \cdot \nabla p_2\, dx \\
&- \int_\Omega \mathbf{K} M_1(s_1) \rho_1(p_1) \mathbf{g} \cdot \nabla p_1\, dx - \int_\Omega \mathbf{K} M_2(s_2) \rho_2(p_2) \mathbf{g} \cdot \nabla p_2\, dx \\
&+ \int_\Omega \rho_1(p_1) s_1 f_p g_1(p_1)\, dx + \int_\Omega \rho_2(p_2) s_2 f_p g_2(p_2)\, dx \\
&= \int_\Omega \rho_1(p_1) s_1^I f_I g_1(p_1)\, dx + \int_\Omega \rho_2(p_2) s_2^I f_I g_2(p_2)\, dx.
\end{aligned} \tag{1.18}$$

A key point is to obtain formally the first term in the above equality, for that let

$$\begin{aligned}
D &= \partial_t(\rho_1(p_1) s_1) g_1(p_1) + \partial_t(\rho_2(p_2) s_2) g_2(p_2) \\
&= \partial_t(\rho_1(p_1) s_1 g_1(p_1)) + \partial_t(\rho_2(p_2) s_2 g_2(p_2)) - s_1 \partial_t p_1 - s_2 \partial_t p_2.
\end{aligned}$$

We have $s_1 + s_2 = 1$, then $s_1 \partial_t p_1 + s_2 \partial_t p_2 = s_1 \partial_t f(s_1) + \partial_t p_2 = \partial_t G(s_1) + \partial_t p_2$, where G is a primitive of $s_1 f'(s_1)$. We can write D as $D = \partial_t E$ where E is defined by

$$\begin{aligned}
E &= \rho_1(p_1) s_1 g_1(p_1) + \rho_2(p_2) s_2 g_2(p_2) - G(s_1) - p_2 \\
&= s_1(\rho_1(p_1) g_1(p_1) - p_1) + s_2(\rho_2(p_2) s_2 g_2(p_2) - p_2) - G(s_1) + s_1 f(s_1),
\end{aligned}$$

from the definition of the functions $\mathcal{H}_i$ ($i = 1, 2$) and G, the expression of E is equivalent to :

$$E = s_1 \mathcal{H}_1(p_1) + s_2 \mathcal{H}_2(p_2) + \int_0^{s_1} f(\xi)\, d\xi.$$

Using the assumptions (H1)–(H6) and the fact that $\mathcal{H}_i \geq 0$, $g_i(p_i)$ is sublinear with respect to p_i we deduce from (1.18) that

$$\int_{Q_T} M_1(s_1)\nabla p_1 \cdot \nabla p_1 \, dx + \int_{Q_T} M_2(s_2)\nabla p_2 \cdot \nabla p_2 \, dx < \infty, \tag{1.19}$$

By definition of global pressure, we have

$$\nabla p = \nabla p_2 + \frac{M_1}{M}\nabla f(s_1) = \nabla p_1 - \frac{M_2}{M}\nabla f(s_1), \tag{1.20}$$

then, we deduce a magic equality

$$\begin{aligned} \int_{Q_T} M|\nabla p|^2 \, dx + \int_{Q_T} \frac{M_1 M_2}{M}|\nabla f(s_1)|^2 \, dx \\ = \int_{Q_T} M_1(s_1)|\nabla p_1|^2 \, dx + \int_{Q_T} M_2(s_2)|\nabla p_2|^2 \, dx, \end{aligned} \tag{1.21}$$

thus, the equality (1.21) and the assumption (H3) ensure that $p \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$ and $\beta(s_1) \in L^2(0,T;H^1(\Omega))$.

Before establishing theorem 3.1, we introduce the existence of solutions to system (1.1) under the assumptions (H1)-(H7), with the addition of some terms on each equation to save a maximum principle on saturations, to conserve the existence of solutions of a time discretization , and to insure a compactness *lemma* which is necessary to pass from an elliptic problem to parabolic one, after that we get rid of these terms by a limit process which is also conserved. Thus we consider the non-degenerate system :

$$\begin{aligned} \phi\partial_t(\rho_1(p_1^\eta)s_1^\eta) - \operatorname{div}(\mathbf{K}\rho_1(p_1^\eta)M_1(s_1^\eta)\nabla p_1^\eta) + \operatorname{div}(\mathbf{K}\rho_1^2(p_1^\eta)M_1(s_1^\eta)\mathbf{g}) \\ - \eta \operatorname{div}(\rho_1(p_1^\eta)\nabla(p_1^\eta - p_2^\eta)) + \rho_1(p_1^\eta)s_1^\eta f_P = \rho_1(p_1^\eta)s_1^I f_I, \end{aligned} \tag{1.22}$$

$$\begin{aligned} \phi\partial_t(\rho_2(p_2^\eta)s_2^\eta) - \operatorname{div}(\mathbf{K}\rho_2(p_2^\eta)M_2(s_2^\eta)\nabla p_2^\eta) + \operatorname{div}(\mathbf{K}\rho_2^2(p_2^\eta)M_2(s_2^\eta)\mathbf{g}) \\ - \eta \operatorname{div}(\rho_2(p_2^\eta)\nabla(p_2^\eta - p_1^\eta)) + \rho_2(p_2^\eta)s_2^\eta f_P = \rho_2(p_2^\eta)s_2^I f_I, \end{aligned} \tag{1.23}$$

completed with the initial conditions (1.9), and the following mixed boundary conditions,

$$\begin{cases} p_1^\eta(t,x) = 0, \;\; p_2^\eta(t,x) = 0 & \text{on } \Gamma_1 \\ \Big(-\mathbf{K}M_1(s_1^\eta)(\nabla p_1^\eta - \rho_1(p_1^\eta)\mathbf{g}) - \eta\nabla(p_1^\eta - p_2^\eta)\Big) \cdot \mathbf{n} = 0 & \text{on } \Gamma_{imp} \\ \Big(-\mathbf{K}M_2(s_2^\eta)(\nabla p_2^\eta - \rho_2(p_2^\eta)\mathbf{g}) + \eta\nabla(p_1^\eta - p_2^\eta)\Big) \cdot \mathbf{n} = 0 & \text{on } \Gamma_{imp} \end{cases} \tag{1.24}$$

where $\mathbf{n}$ is the outward normal to the boundary Γ_{imp}.

Now, we state existence of solutions of the above system by the following theorem.
Theorem 3.2. *(**Non-degenerate system**) Let (H1)-(H6) hold. Let $(p_1^0,\, p_2^0)$ belongs to $L^2(\Omega)\times L^2(\Omega)$. Then for all $\eta>0$, there exists (p_1^η, p_2^η) satisfying*

$$
\begin{aligned}
&p_i^\eta \in L^2(0,T;H^1_{\Gamma_1}(\Omega)),\ s_1^\eta \in L^2(0,T;H^1(\Omega)),\ s_2^\eta \in L^2(0,T;H^1_{\Gamma_1}(\Omega)),\\
&\phi\partial_t(\rho_i(p_i^\eta)s_i^\eta) \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))'),\ \rho_i(p_i^\eta)s_i^\eta) \in C^0(0,T;L^2(\Omega)),\\
&0\le s_i^\eta(t,x)\le 1 \ a.e\ in\ Q_T,\ i=1,2,
\end{aligned}
$$

for all $\varphi,\xi \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$,

$$
\begin{aligned}
&\langle \phi\partial_t(\rho_1(p_1^\eta)s_1^\eta),\varphi\rangle + \int_{Q_T} \mathbf{K}M_1(s_1^\eta)\rho_1(p_1^\eta)\nabla p_1^\eta\cdot\nabla\varphi\,dxdt\\
&-\int_{Q_T}\mathbf{K}M_1(s_1^\eta)\rho_1^2(p_1^\eta)\mathbf{g}\cdot\nabla\varphi\,dxdt + \eta\int_{Q_T}\rho_1(p_1^\eta)\nabla(p_1^\eta-p_2^\eta)\cdot\nabla\varphi\,dxdt\\
&+\int_{Q_T}\rho_1(p_1^\eta)s_1^\eta f_P\varphi\,dxdt = \int_{Q_T}\rho_1(p_1^\eta)s_1^I f_I\varphi\,dxdt
\end{aligned}
\tag{1.25}
$$

$$
\begin{aligned}
&\langle \phi\partial_t(\rho_2(p_2^\eta)s_2^\eta),\xi\rangle + \int_{Q_T} \mathbf{K}M_2(s_2^\eta)\rho_2(p_2^\eta)\nabla p_2^\eta\cdot\nabla\xi\,dxdt\\
&-\int_{Q_T}\mathbf{K}M_2(s_2^\eta)\rho_2^2(p_2^\eta)\mathbf{g}\cdot\nabla\xi\,dxdt - \eta\int_{Q_T}\rho_2(p_2^\eta)\nabla(p_1^\eta-p_2^\eta)\cdot\nabla\xi\,dxdt\\
&+\int_{Q_T}\rho_2(p_2^\eta)s_2^\eta f_P\xi\,dxdt = \int_{Q_T}\rho_2(p_2^\eta)s_2^I f_I\xi\,dxdt
\end{aligned}
\tag{1.26}
$$

where the bracket $\langle\cdot,\cdot\rangle$ represents the duality product between $L^2(0,T;(H^1_{\Gamma_1}(\Omega))')$ and $L^2(0,T;H^1_{\Gamma_1}(\Omega))$.

Before establishing theorems 3.2 and 3.1, we introduce the existence of solutions to a time discretization of (1.22)-(1.23),

$$
\begin{aligned}
\phi\frac{\rho_1(p_1)s_1-\rho_1^\star s_1^\star}{h} &- \operatorname{div}(\mathbf{K}\rho_1(p_1)M_1(s_1)\nabla p_1) + \operatorname{div}(\mathbf{K}M_1(s_1)\rho_1^2\mathbf{g})\\
&- \eta\operatorname{div}(\rho_1(p_1)\nabla(p_1-p_2)) + \rho_1(p_1)s_1f_P = \rho_1(p_1)s_1^If_I,
\end{aligned}
$$

$$
\begin{aligned}
\phi\frac{\rho_2(p_2)s_2-\rho_2^\star s_2^\star}{h} &- \operatorname{div}(\mathbf{K}\rho_2(p_2)M_2(s_2)\nabla p_2) + \operatorname{div}(\mathbf{K}M_2(s_2)\rho_2^2\mathbf{g})\\
&- \eta\operatorname{div}(\rho_2(p_2)\nabla(p_2-p_1)) + \rho_2(p_2)s_2f_P = \rho_2(p_2)s_2^If_I,
\end{aligned}
$$

where $\rho_i^\star$ and $s_i^\star$, formally, are the values of the $h-$translated in time of $\rho_i(p_i)$ and s_i respectively, $i=1,2$.
Existence of solutions of the above system is given in the following theorem.
Theorem 3.3. *(**Non-degenerate elliptic system**) Let (H1)-(H6) hold. Let $(p_1^0,\, p_2^0)$ belongs to $L^2(\Omega)\times L^2(\Omega)$. Then for all $h>0$, there exists $(p_1^h, p_2^h) =$*

$(p_1^{\eta,h}, p_2^{\eta,h})$ *satisfying*

$$p_1^h \in H^1_{\Gamma_1}(\Omega),\ p_2^h \in H^1_{\Gamma_1}(\Omega),\ s_1^h \in H^1(\Omega),\ s_2^h \in H^1_{\Gamma_1}(\Omega),$$
$$0 \le s_i^h(t,x) \le 1 \ a.e \ in \ Q_T,\ i = 1,2,$$

for all $\varphi, \xi \in H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \phi \frac{\rho_1(p_1^h)s_1^h - \rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \mathbf{K}M_1(s_1^h)\rho_1(p_1^h)\nabla p_1^h \cdot \nabla\varphi\,dx$$
$$- \int_\Omega \mathbf{K}M_1(s_1^h)\rho_1^2(p_1^h)\mathbf{g}\cdot\nabla\varphi\,dx + \eta\int_\Omega \rho_1(p_1^h)\nabla(p_1^h - p_2^h)\cdot\nabla\varphi\,dx$$
$$+ \int_\Omega \rho_1(p_1^h)s_1^h f_P\varphi\,dx = \int_\Omega \rho_1(p_1^h)s_1^I f_I\varphi\,dx, \tag{1.27}$$

$$\int_\Omega \phi \frac{\rho_2(p_2^h)s_2^h - \rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \mathbf{K}M_2(s_2^h)\rho_2(p_2^h)\nabla p_2^h \cdot \nabla\xi\,dx$$
$$- \int_\Omega \mathbf{K}M_2(s_2^h)\rho_2^2(p_2^h)\mathbf{g}\cdot\nabla\xi\,dx - \eta\int_\Omega \rho_2(p_2^h)\nabla(p_1^h - p_2^h)\cdot\nabla\xi\,dx$$
$$+ \int_\Omega \rho_2(p_2^h)s_2^h f_P\xi\,dx = \int_\Omega \rho_2(p_2^h)s_2^I f_I\xi\,dx, \tag{1.28}$$

The rest of the paper is organized as follows. In the next section we deal with the time discrete model to prove theorem 3.3 in two steps. The first step deals with an elliptic system with non degenerate mobilities, $M_i^\varepsilon = M_i + \varepsilon$ with $\varepsilon > 0$, in this step we apply a suitable fixed point theorem, Leray-Schauder, to get weak solution. The second step is to pass to the limit as ε goes to zero depending on a suitable uniform estimate (w. r. to ε), and a maximum principle ensures the positivity of saturations which achieves the proof of theorem 3.3.
In the third section we introduce a sequence of solutions solving (1.27) (1.28). This choice is motivated by the fact that no evolution have to be considered in a first step. The problem of degeneracy of evolution term is temporarily sat aside. Furthermore, the maximum principle is conserved on saturation after the passage to the limit on in the non linear variational elliptic system. The last section is devoted to pass from non-degenerate case to degenerate case through a compactness *lemma* which allow us with the help of some estimates to pass the limit and end the proof of existence of weak solutions of the system under consideration.
The next section is devoted to the analysis of the elliptic problem.

2 Study of a nonlinear elliptic system (proof of theorem 3.3)

Having in mind a time discretization of (1.22)-(1.23), we are concerned with the following system,

$$\phi\frac{\rho_1(p_1)s_1-\rho_1^\star s_1^\star}{h}-\operatorname{div}(\mathbf{K}\rho_1(p_1)M_1(s_1)\nabla p_1)+\operatorname{div}(\mathbf{K}M_1(s_1)\rho_1^2\mathbf{g})$$
$$-\eta\operatorname{div}(\rho_1(p_1)\nabla(p_1-p_2))+\rho_1(p_1)s_1f_P=\rho_1(p_1)s_1^If_I,\quad (2.1)$$

$$\phi\frac{\rho_2(p_2)s_2-\rho_2^\star s_2^\star}{h}-\operatorname{div}(\mathbf{K}\rho_2(p_2)M_2(s_2)\nabla p_2)+\operatorname{div}(\mathbf{K}M_2(s_2)\rho_2^2\mathbf{g})$$
$$-\eta\operatorname{div}(\rho_2(p_2)\nabla(p_2-p_1))+\rho_2(p_2)s_2f_P=\rho_2(p_2)s_2^If_I,\quad (2.2)$$

where $\rho_i^\star$ and $s_i^\star$, formally, are the values of the $h-$translated in time of $\rho_i(p_i)$ and s_i respectively, $i=1,2$.
Before establishing theorem 3.3 which is the main purpose of this section, we introduce the existence of solutions of system (2.1)(2.2), when the mobilities $M_i,\ (i=1,2.)$, are replaced by a non-degenerate positive functions,

$$M_i^\varepsilon=M_i+\varepsilon,\quad i=1,2,\ \text{and } \varepsilon>0,$$

which reinforce the passage to the limit in another regularization which is the trunk high frequencies of nonlinear elliptic term in pressure p_2 in the equation (2.1). Let $\mathcal{P}_N$ be the orthogonal projector of $L^2(\Omega)$ on the first N eigenvectors of the operator

$$p\longrightarrow-\Delta p$$

with homogeneous Dirichlet boundary conditions.
The projector $\mathcal{P}_N$ appears in (2.4) to make regular the implied term. The necessity of this regularization appears in the coming proposition in order to define the operator which we apply on the Leray-Schauder fixed point theorem.

The addition of such ε to the mobilities lead to the loss of maximum principle on the saturations s_i $(i=1,2.)$ so the functions M_1 and M_2 are extended on $\mathbb{R}$ by continues constant functions outside $[0,1]$ and then are bounded on $\mathbb{R}$. For the same reason we denote,

$$Z(s_i)=\begin{cases}0 & \text{for } s_i\le 0\\ s_i & \text{for } s_i\in[0;1]\\ 1 & \text{for } s_i\ge 1.\end{cases}\quad (2.3)$$

In the same spirit and in order to write the saturations s_i $(i = 1, 2.)$ as functions of the principle unknowns p_1 and p_2 of the system, we extend the capillary pressure function f by continuity and strict monotony outside $[0, 1]$ in to $\bar{f}$, this is possible in the case when the capillary function f is bounded, in other wards when $| f(0) |< \infty$, and denote by $s_1 = \bar{f}^{-1}(p_1 - p_2)$ and $s_2 = 1 - \bar{f}^{-1}(p_1 - p_2)$.
Existence of solution to (2.1)-(2.2) is constructed in three steps. The first one consists in studying the following problem for fixed parameters $\varepsilon > 0$, $N > 0$ and $\eta > 0$. Then, we are concerned with the regularized elliptic system :

$$\begin{aligned}
&\int_\Omega \phi \frac{\rho_1(p_1^{\varepsilon,N}) Z(s_1^{\varepsilon,N}) - \rho_1^\star s_1^\star}{h} \varphi \, dx + \int_\Omega \mathbf{K} M_1^\varepsilon(s_1^{\varepsilon,N}) \rho_1(p_1^{\varepsilon,N}) \nabla p_1^{\varepsilon,N} \cdot \nabla \varphi \, dx \\
&- \int_\Omega \mathbf{K} M_1(s_1^{\varepsilon,N}) \rho_1^2(p_1^{\varepsilon,N}) \mathbf{g} \cdot \nabla \varphi \, dx + \eta \int_\Omega \rho_1(p_1^{\varepsilon,N}) \nabla(\mathcal{P}_N p_1^{\varepsilon,N} - \mathcal{P}_N p_2^{\varepsilon,N}) \cdot \nabla \varphi \, dx \\
&+ \int_\Omega \rho_1(p_1^{\varepsilon,N}) Z(s_1^{\varepsilon,N}) f_P \varphi \, dx = \int_\Omega \rho_1(p_1^{\varepsilon,N}) s_1^I f_I \varphi \, dx, \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \phi \frac{\rho_2(p_2^{\varepsilon,N}) Z(s_2^{\varepsilon,N}) - \rho_2^\star s_2^\star}{h} \xi \, dx + \int_\Omega \mathbf{K} M_2^\varepsilon(s_2^{\varepsilon,N}) \rho_2(p_2^{\varepsilon,N}) \nabla p_2^{\varepsilon,N} \cdot \nabla \xi \, dx \\
&- \int_\Omega \mathbf{K} M_2(s_2^{\varepsilon,N}) \rho_2^2(p_2^{\varepsilon,N}) \mathbf{g} \cdot \nabla \xi \, dx - \eta \int_\Omega \rho_2(p_2^{\varepsilon,N}) \nabla(\mathcal{P}_N p_1^{\varepsilon,N} - \mathcal{P}_N p_2^{\varepsilon,N}) \cdot \nabla \xi \, dx \\
&+ \int_\Omega \rho_2(p_2^{\varepsilon,N}) Z(s_2^{\varepsilon,N}) f_P \xi \, dx = \int_\Omega \rho_2(p_2^{\varepsilon,N}) s_2^I f_I \xi \, dx, \qquad (2.5)
\end{aligned}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.
The second step concerns the passage to the limit as N goes to infinity in order to recover the full physical diffusion on pressures p_1 and p_2, while the third one is the passage to the limit as ε goes to zero.

Step 1. We show for fixed $N > 0$ and $\varepsilon > 0$ existence of solutions to (2.4)-(2.5). We omit for the time being the dependence of solutions on parameter $N > 0$ and ε.
Proposition 3.1. *Assume $\rho_i^\star s_i^\star$ belongs to $L^2(\Omega)$ and $\rho_i^\star s_i^\star \geq 0$. Then there exists (p_1, p_2) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, solution of (2.4)-(2.5).*

Démonstration. The proof is based on the Leray-Schauder fixed point theorem. Let $\mathcal{T}$ be a map from $L^2(\Omega) \times L^2(\Omega)$ to $L^2(\Omega) \times L^2(\Omega)$ defined by

$$\mathcal{T}(\overline{p}_1, \overline{p}_2) = (p_1, p_2),$$

where the pair (p_1, p_2) is the unique solution of the system (2.6)-(2.7)

$$\int_\Omega \phi \frac{\rho_1(\overline{p}_1)Z(\overline{s}_1) - \rho_1^\star s_1^\star}{h}\varphi\, dx + \int_\Omega \mathbf{K}M_1^\varepsilon(\overline{s}_1)\rho_1(\overline{p}_1)\nabla p_1 \cdot \nabla\varphi\, dx$$
$$- \int_\Omega \mathbf{K}M_1(\overline{s}_1)\rho_1^2(\overline{p}_1)\mathbf{g}\cdot\nabla\varphi\, dx + \eta\int_\Omega \rho_1(\overline{p}_1)\nabla(\mathcal{P}_N\overline{p}_1 - \mathcal{P}_N\overline{p}_2)\cdot\nabla\varphi\, dx$$
$$+ \int_\Omega \rho_1(\overline{p}_1)Z(\overline{s}_1)f_P\varphi\, dx = \int_\Omega \rho_1(\overline{p}_1)s_1^I f_I\varphi\, dx, \tag{2.6}$$

$$\int_\Omega \phi \frac{\rho_2(\overline{p}_2)Z(\overline{s}_2) - \rho_2^\star s_2^\star}{h}\xi\, dx + \int_\Omega \mathbf{K}M_2^\varepsilon(\overline{s}_2)\rho_2(\overline{p}_2)\nabla p_2 \cdot \nabla\xi\, dx$$
$$- \int_\Omega \mathbf{K}M_2(\overline{s}_2)\rho_2^2(\overline{p}_2)\mathbf{g}\cdot\nabla\xi\, dx - \eta\int_\Omega \rho_2(\overline{p}_2)\nabla(\mathcal{P}_N\overline{p}_1 - \mathcal{P}_N\overline{p}_2)\cdot\nabla\xi\, dx$$
$$+ \int_\Omega \rho_2(\overline{p}_2)Z(\overline{s}_2)f_P\xi\, dx = \int_\Omega \rho_2(\overline{p}_2)s_2^I f_I\xi\, dx, \tag{2.7}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$, $\overline{s}_1 = \bar{f}^{-1}(\overline{p}_1 - \overline{p}_2)$ and $\overline{s}_2 = 1 - \bar{f}^{-1}(\overline{p}_1 - \overline{p}_2)$. The functions M_1 and M_2 are the extended mobilities which operates on $\mathbb{R}$. Such extensions of the mobilities M_i $(i = 1, 2.)$, the capillary function f and such bound of the saturations s_i $(i = 1, 2.)$ by introducing the map Z are temporary; we deal it at the end of this section after the passage to the limit in ε by a maximum principle on saturations and then the mobilities, the map Z and the extended capillary function $\bar{f}$ operates only on $[0, 1]$ where they have a physical meaning.

The system (2.6)−(2.7) can be written under the form $B_1(p_1, \varphi) = f_1(\varphi)$, $B_2(p_2, \xi) = f_2(\xi)$, where

$$B_1(p_1, \varphi) = \int_\Omega \mathbf{K}M_1^\varepsilon(\overline{s}_1)\rho_1(\overline{p}_1)\nabla p_1\cdot\nabla\varphi\, dx,$$

$$f_1(\varphi) = -\int_\Omega \phi \frac{\rho_1(\overline{p}_1)Z(\overline{s}_1) - \rho_1^\star s_1^\star}{h}\varphi\, dx + \int_\Omega \mathbf{K}M_1(\overline{s}_1)\rho_1^2(\overline{p}_1)\mathbf{g}\cdot\nabla\varphi\, dx$$
$$- \int_\Omega \rho_1(\overline{p}_1)Z(\overline{s}_1)f_P\varphi\, dx + \int_\Omega \rho_1(\overline{p}_1)s_1^I f_I\varphi\, dx$$
$$- \eta\int_\Omega \rho_1(\overline{p}_1)\nabla(\mathcal{P}_N\overline{p}_1 - \mathcal{P}_N\overline{p}_2)\cdot\nabla\varphi\, dx,$$

$$B_2(p_2, \xi) = \int_\Omega \mathbf{K}M_2^\varepsilon(\overline{s}_2)\rho_2(\overline{p}_2)\nabla p_2\cdot\nabla\xi\, dx,$$

$$\begin{aligned}
f_2(\xi) = &- \int_\Omega \phi \frac{\rho_2(\overline{p}_2) Z(\overline{s}_2) - \rho_2^\star s_2^\star}{h} \xi \, dx + \int_\Omega \mathbf{K} M_2(\overline{s}_2) \rho_2^2(\overline{p}_2) \mathbf{g} \cdot \nabla \xi \, dx \\
&- \int_\Omega \rho_2(\overline{p}_2) Z(\overline{s}_2) f_P \xi \, dx + \int_\Omega \rho_2(\overline{p}_2) s_2^I f_I \, \xi \, dx \\
&+ \eta \int_\Omega \rho_2(\overline{p}_2) \nabla(\mathcal{P}_N \overline{p}_1 - \mathcal{P}_N \overline{p}_2) \cdot \nabla \xi \, dx.
\end{aligned}$$

$B_1(\cdot,\cdot)$ and $B_2(\cdot,\cdot)$ are bilinear, continuous and coercive mappings on $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$. $f_1(\cdot)$ and $f_2(\cdot)$ are linear continuous mappings on $H^1_{\Gamma_1}(\Omega)$. Then, apply Lax-Milgram theorem to get the existence of the unique pair (p_1, p_2) in $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ which ensures that the map $\mathcal{T}$ is well defined on $L^2(\Omega) \times L^2(\Omega)$.

Lemma 3.1. *The map $\mathcal{T}$ is a continuous operator which maps every bounded subsets of $L^2(\Omega)$ into a relatively compact set.*

Démonstration. Consider a sequence $(\overline{p}_{1,n}, \overline{p}_{2,n})$ of a bounded set of $L^2(\Omega) \times L^2(\Omega)$ which converges to $(\overline{p}_1, \overline{p}_2) \in L^2(\Omega) \times L^2(\Omega)$, and let us prove that $(p_{1,n}, p_{2,n}) = \mathcal{T}(\overline{p}_{1,n}, \overline{p}_{2,n})$ is bounded in $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ which converges to $(p_1, p_2) = \mathcal{T}(\overline{p}_1, \overline{p}_2)$. The sequences $p_{1,n},\ p_{2,n}$ verify respectively

$$\begin{aligned}
&\int_\Omega \phi \frac{\rho_1(\overline{p}_{1,n}) Z(\overline{s}_{1,n}) - \rho_1^\star s_1^\star}{h} \varphi \, dx + \int_\Omega \mathbf{K} M_1^\varepsilon(\overline{s}_{1,n}) \rho_1(\overline{p}_{1,n}) \nabla p_{1,n} \cdot \nabla \varphi \, dx \\
&- \int_\Omega \mathbf{K} M_1(\overline{s}_{1,n}) \rho_1^2(\overline{p}_{1,n}) \mathbf{g} \cdot \nabla \varphi \, dx + \eta \int_\Omega \rho_1(\overline{p}_{1,n}) \nabla(\mathcal{P}_N \overline{p}_{1,n} - \mathcal{P}_N \overline{p}_{2,n}) \cdot \nabla \varphi \, dx \\
&+ \int_\Omega \rho_1(\overline{p}_{1,n}) Z(\overline{s}_{1,n}) f_P \varphi \, dx = \int_\Omega \rho_1(\overline{p}_{1,n}) s_1^I f_I \, \varphi \, dx,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&\int_\Omega \phi \frac{\rho_2(\overline{p}_{2,n}) Z(\overline{s}_{2,n}) - \rho_2^\star s_2^\star}{h} \xi \, dx + \int_\Omega \mathbf{K} M_2^\varepsilon(\overline{s}_{2,n}) \rho_2(\overline{p}_{2,n}) \nabla p_{2,n} \cdot \nabla \xi \, dx \\
&- \int_\Omega \mathbf{K} M_2(\overline{s}_{2,n}) \rho_2^2(\overline{p}_{2,n}) \mathbf{g} \cdot \nabla \xi \, dx - \eta \int_\Omega \rho_2(\overline{p}_{2,n}) \nabla(\mathcal{P}_N \overline{p}_{1,n} - \mathcal{P}_N \overline{p}_{2,n}) \cdot \nabla \xi \, dx \\
&+ \int_\Omega \rho_2(\overline{p}_{2,n}) Z(\overline{s}_{2,n}) f_P \xi \, dx = \int_\Omega \rho_2(\overline{p}_{2,n}) s_2^I f_I \, \xi \, dx,
\end{aligned} \tag{2.9}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.
Let us take $\varphi = p_{1,n}$ in (2.8),

$$\begin{aligned}
&\int_\Omega \mathbf{K} M_1^\varepsilon(\overline{s}_{1,n}) \rho_1(\overline{p}_{1,n}) \nabla p_{1,n} \cdot \nabla p_{1,n} \, dx = \int_\Omega \mathbf{K} M_1(\overline{s}_{1,n}) \rho_1^2(\overline{p}_{1,n}) \mathbf{g} \cdot \nabla p_{1,n} \, dx \\
&- \eta \int_\Omega \rho_1(\overline{p}_{1,n}) \nabla(\mathcal{P}_N \overline{p}_{1,n} - \mathcal{P}_N \overline{p}_{2,n}) \cdot \nabla p_{1,n} \, dx - \int_\Omega \phi \frac{\rho_1(\overline{p}_{1,n}) Z(\overline{s}_{1,n}) - \rho_1^\star s_1^\star}{h} p_{1,n} \, dx \\
&- \int_\Omega \rho_1(\overline{p}_{1,n}) Z(\overline{s}_{1,n}) f_P p_{1,n} \, dx \int_\Omega \rho_1(\overline{p}_{1,n}) s_1^I f_I \, p_{1,n} \, dx,
\end{aligned} \tag{2.10}$$

we deduce from the Cauchy-Schwarz inequality that (2.10) reduces to,

$$\varepsilon k_0 \rho_m \int_\Omega |\nabla p_{1,n}|^2\, dx \le C\Big(1 + \|p_{1,n}\|_{L^2(\Omega)} \\ + \|\nabla p_{1,n}\|_{L^2(\Omega)} + \|\nabla \mathcal{P}_N \overline{p}_{1,n}\|_{L^2(\Omega)} + \|\nabla \mathcal{P}_N \overline{p}_{2,n}\|_{L^2(\Omega)}\Big), \quad (2.11)$$

where C depends on Ω, η, h, ϕ_1, $\|f_P\|_{L^2(\Omega)}$, $\|f_I\|_{L^2(\Omega)}$, ρ_M, k_∞ and $\|\rho_1^\star s_1^\star\|_{L^2(\Omega)}$. As,

$$\|\nabla \mathcal{P}_N \overline{p}_{i,n}\|_{L^2(\Omega)} \le c_N \|\overline{p}_{i,n}\|_{L^2(\Omega)}, \quad (i = 1, 2)$$

where c_N is the square root of the n^{th} eigenvalue of the laplace operator (by considering the set of eigenvalues as increasing sequence), the Poincaré and Young inequalities and the estimate (2.11) ensure that the sequence $(P_{1,n})_n$ is uniformly bounded in $H^1_{\Gamma_1}(\Omega)$.

Then, taking $\xi = p_{2,n}$ in (2.9), we deduce similarly that,

$$\varepsilon k_0 \rho_m \int_\Omega |\nabla p_{2,n}|^2\, dx \le C\Big(1 + \|p_{2,n}\|_{L^2(\Omega)} + \\ \|\nabla p_{2,n}\|_{L^2(\Omega)} + \|\nabla \mathcal{P}_N \overline{p}_{1,n}\|_{L^2(\Omega)} + \|\nabla \mathcal{P}_N \overline{p}_{2,n}\|_{L^2(\Omega)}\Big), \quad (2.12)$$

where C depends on Ω, η, h, ϕ_1, $\|f_P\|_{L^2(\Omega)}$, $\|f_I\|_{L^2(\Omega)}$, ρ_M, k_∞ and $\|\rho_2^\star s_2^\star\|_{L^2(\Omega)}$. Then the sequence $(P_{2,n})_n$ is uniformly bounded in $H^1_{\Gamma_1}(\Omega)$. This establishes the relative compactness property of the map $\mathcal{T}$.

Furthermore, up to a subsequence, we have the convergences

$$p_{1,n} \longrightarrow p_1 \quad \text{weakly in } H^1_{\Gamma_1}(\Omega), \quad (2.13)$$
$$p_{2,n} \longrightarrow p_2 \quad \text{weakly in } H^1_{\Gamma_1}(\Omega), \quad (2.14)$$
$$p_{1,n} \longrightarrow p_1 \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \quad (2.15)$$
$$p_{2,n} \longrightarrow p_2 \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \quad (2.16)$$

In order to complete the proof of continuity of the operator $\mathcal{T}$, it is enough to show that (p_1, p_2) is the unique adherent value of the sequence $(p_{1,n}, p_{2,n})$, for that let us show (p_1, p_2) is the unique solution of (2.6)-(2.7) by passing the limit in (2.8)-(2.9).

Passage to the limit in (2.8) :

$$\int_\Omega \phi \frac{\rho_1(\overline{p}_{1,n})Z(\overline{s}_{1,n}) - \rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \mathbf{K}M_1^\varepsilon(\overline{s}_{1,n})\rho_1(\overline{p}_{1,n})\nabla p_{1,n}\cdot\nabla\varphi\,dx$$
$$-\int_\Omega \mathbf{K}M_1(\overline{s}_{1,n})\rho_1^2(\overline{p}_{1,n})\mathbf{g}\cdot\nabla\varphi\,dx + \eta\int_\Omega \rho_1(\overline{p}_{1,n})\nabla(\mathcal{P}_N\overline{p}_{1,n} - \mathcal{P}_N\overline{p}_{2,n})\cdot\nabla\varphi\,dx$$
$$+\int_\Omega \rho_1(\overline{p}_{1,n})Z(\overline{s}_{1,n})f_P\varphi\,dx = \int_\Omega \rho_1(\overline{p}_{1,n})s_1^I f_I\varphi\,dx,$$

where $\overline{s}_{1,n} = \bar{f}^{-1}(\overline{p}_{1,n} - \overline{p}_{2,n})$.
The passage to the limit in the first term is due to the continuity of Z, $\bar{f}^{-1}$ and ρ_1, the convergences (2.15) and (2.16), and the domination of $\rho_1(\overline{p}_{1,n})Z(\overline{s}_{1,n})\varphi$ by $\rho_M|\varphi|$, which allow us to apply the Lebesgue theorem.
The second term is treated as follows, the sequence $\left(KM_1^\varepsilon(\overline{s}_{1n})\rho_1(\overline{p}_{1n})\nabla\varphi\right)_n$ is dominated and converges a.e. as n goes to infinity. Then, by Lebesgue theorem, we have the following strong convergence in $L^2(\Omega)$,

$$KM_1^\varepsilon(\overline{s}_{1,n})\rho_1(\overline{p}_{1n})\nabla\varphi \longrightarrow KM_1^\varepsilon(\overline{s}_1)\rho_1(\overline{p}_1)\nabla\varphi. \tag{2.17}$$

Furthermore and due to the convergence (2.13), it follows that

$$\nabla p_1^n \longrightarrow \nabla p_1 \qquad \text{weakly in } L^2(\Omega), \tag{2.18}$$

then, the convergences (2.19) and (2.20) establish the limit for the second term.
The fourth term

$$\eta\int_\Omega \rho_1(\overline{p}_{1,n})\nabla(\mathcal{P}_N\overline{p}_{1,n} - \mathcal{P}_N\overline{p}_{2,n})\cdot\nabla\varphi\,dx,$$

is treated as follows,

$$\rho_i(\overline{p}_{i,n})\nabla\varphi \longrightarrow \rho_i(\overline{p}_i)\nabla\varphi \text{ strongly in } (L^2(\Omega))^d \;\; (i=1,2). \tag{2.19}$$

Furthermore $\overline{p}_{i,n}$ converges in $L^2(\Omega)$, it follows that

$$\nabla\mathcal{P}_N\overline{p}_{i,n} \longrightarrow \nabla\mathcal{P}_N\overline{p}_i \text{ strongly in } (L^2(\Omega))^d \;\; (i=1,2). \tag{2.20}$$

Then, the convergences (2.19)-(2.20) allow us to pass the limit in the fourth term. The convergences of the other terms are always an application of the Lebesgue convergence theorem.
The passage to the limit on (2.9) is obtained in the same manner. Thus (p_1, p_2) is a solution of (2.6)-(2.7), which establishes the continuity and achieves the proof of the lemma. □

Lemma 3.2. *(A priori estimate) There exists $r > 0$ such that, if $(p_1, p_2) = \lambda\mathcal{T}(p_1, p_2)$ with $\lambda \in (0, 1)$, then*

$$\|(p_1, p_2)\|_{L^2(\Omega)\times L^2(\Omega)} \le r.$$

Démonstration. Assume $(p_1, p_2) = \lambda\mathcal{T}(p_1, p_2)$ holds , then (p_1, p_2) satisfies

$$\begin{aligned}\int_\Omega \mathbf{K}M_1^\varepsilon(s_1)\rho_1(p_1)\nabla p_1\cdot\nabla\varphi\,dx &= -\lambda\int_\Omega \phi\frac{\rho_1(p_1)Z(s_1)-\rho_1^\star s_1^\star}{h}\varphi\,dx\\ +\lambda\int_\Omega \mathbf{K}M_1(s_1)\rho_1^2(p_1)\mathbf{g}\cdot\nabla\varphi\,dx &- \lambda\int_\Omega\rho_1(p_1)Z(s_1)f_P\varphi\,dx + \lambda\int_\Omega\rho_1(p_1)s_1^I f_I\varphi\,dx\\ &- \lambda\eta\int_\Omega\rho_1(p_1)\nabla(\mathcal{P}_Np_1-\mathcal{P}_Np_2)\cdot\nabla\varphi\,dx,\end{aligned}\tag{2.21}$$

$$\begin{aligned}\int_\Omega \mathbf{K}M_2^\varepsilon(s_2)\rho_2(p_2)\nabla p_2\cdot\nabla\xi\,dx &= -\lambda\int_\Omega \phi\frac{\rho_2(p_2)Z(s_2)-\rho_2^\star s_2^\star}{h}\xi\,dx\\ +\lambda\int_\Omega \mathbf{K}M_2(s_2)\rho_2^2(p_2)\mathbf{g}\cdot\nabla\xi\,dx &- \lambda\int_\Omega\rho_2(p_2)Z(s_2)f_P\xi\,dx + \lambda\int_\Omega\rho_2(p_2)s_2^I f_I\xi\,dx\\ &+ \lambda\eta\int_\Omega\rho_2(p_2)\nabla(\mathcal{P}_Np_1-\mathcal{P}_Np_2)\cdot\nabla\xi\,dx.\end{aligned}\tag{2.22}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$.
Consider $\varphi = g_1(p_1) := \int_0^{p_1}\frac{1}{\rho_1(\zeta)}\,d\zeta \in H^1_{\Gamma_1}(\Omega)$ in (2.21) and $\xi = g_2(p_2) := \int_0^{p_2}\frac{1}{\rho_2(\zeta)}\,d\zeta \in H^1_{\Gamma_1}(\Omega)$ in (2.22). Summing these quantities, we obtain

$$\begin{aligned}\lambda\int_\Omega\frac{\phi}{h}\Big(\big(\rho_1(p_1)Z(s_1)-\rho_1^\star s_1^\star\big)g_1(p_1)+\big(\rho_2(p_2)Z(s_2)-\rho_2^\star s_2^\star\big)g_2(p_2)\Big)\,dx\\ +\int_\Omega\mathbf{K}M_1^\varepsilon\nabla p_1\cdot\nabla p_1\,dx+\lambda\eta\int_\Omega\nabla(\mathcal{P}_Np_1-\mathcal{P}_Np_2)\cdot\nabla(p_1-p_2)\,dx\\ -\lambda\int_\Omega\mathbf{K}\rho_1(p_1)M_1(s_1)g\cdot\nabla p_1\,dx+\int_\Omega\mathbf{K}M_2^\varepsilon\nabla p_2\cdot\nabla p_2\,dx\\ -\lambda\int_\Omega\mathbf{K}\rho_2(p_2)M_2(s_2)g\cdot\nabla p_2\,dx+\lambda\int_\Omega\Big(\rho_1(p_1)Z(s_1)g_1(p_1)+\rho_2(p_2)Z(s_2)g_2(p_2)\Big)f_P\,dx\\ =\lambda\int_\Omega\Big((\rho_1(p_1)s_1^Ig_1(p_1)+\rho_2(p_2)s_2^Ig_2(p_2)\Big)f_I\,dx.\end{aligned}\tag{2.23}$$

Remark that the functions $p_i \to g_i(p_i)$ is sub-linear, we deduce from Cauchy-Schwarz and Poincaré inequalities that (2.23) reduces to

$$\begin{aligned}\varepsilon\int_\Omega|\nabla p_1|^2\,dx+\varepsilon\int_\Omega|\nabla p_2|^2\,dx+\lambda\eta\int_\Omega|\nabla(\mathcal{P}_Np_1-\mathcal{P}_Np_2)|^2\,dx\\ \le C_1(1+\|f_P\|^2_{L^2(\Omega)}+\|f_I\|^2_{L^2(\Omega)}+\|\rho_1^\star s_1^\star\|^2_{L^2(\Omega)}+\|\rho_2^\star s_2^\star\|^2_{L^2(\Omega)}),\end{aligned}\tag{2.24}$$

where C_1 depends on ε and not on λ. □

Lemma 3.1, Lemma 3.2 allow to apply the Leray-Schauder fixed point theorem [66], thus the proof of proposition 3.1 is completed. □

Step 2. Now we are concerned with the limit N goes to infinity (we omit the dependence of solutions on ε). For all N, we have established a solution $(p_{1,N}, p_{2,N}) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ to (2.4) (2.5) satisfying

$$\begin{aligned} \int_\Omega \phi \frac{\rho_1(p_1^N)Z(s_1^N) - \rho_1^\star s_1^\star}{h}\varphi\, dx + \int_\Omega \mathbf{K} M_1^\varepsilon(s_1^N)\rho_1(p_1^N)\nabla p_1^N \cdot \nabla\varphi\, dx \\ - \int_\Omega \mathbf{K} M_1(s_1^N)\rho_1^2(p_1^N)\mathbf{g}\cdot\nabla\varphi\, dx + \eta\int_\Omega \rho_1(p_1^N)\nabla(\mathcal{P}_N p_1^N - \mathcal{P}_N p_2^N)\cdot\nabla\varphi\, dx \\ + \int_\Omega \rho_1(p_1^N)Z(s_1^N)f_P\varphi\, dx = \int_\Omega \rho_1(p_1^N)s_1^I f_I\varphi\, dx, \end{aligned} \tag{2.25}$$

$$\begin{aligned} \int_\Omega \phi \frac{\rho_2(p_2^N)Z(s_2^N) - \rho_2^\star s_2^\star}{h}\xi\, dx + \int_\Omega \mathbf{K} M_2^\varepsilon(s_2^N)\rho_2(p_2^N)\nabla p_2^N \cdot \nabla\xi\, dx \\ - \int_\Omega \mathbf{K} M_2(s_2^N)\rho_2^2(p_2^N)\mathbf{g}\cdot\nabla\xi\, dx - \eta\int_\Omega \rho_2(p_2^N)\nabla(\mathcal{P}_N p_1^N - \mathcal{P}_N p_2^N)\cdot\nabla\xi\, dx \\ + \int_\Omega \rho_2(p_2^N)Z(s_2^N)f_P\xi\, dx = \int_\Omega \rho_2(p_2^N)s_2^I f_I\xi\, dx, \end{aligned} \tag{2.26}$$

for all (φ, ξ) belonging to $H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$.
Reproducing the estimate (2.24) with $\lambda = 1$, we get

$$\begin{aligned} \varepsilon\int_\Omega |\nabla p_1|^2\, dx + \varepsilon\int_\Omega |\nabla p_2|^2\, dx + \eta\int_\Omega |\nabla(\mathcal{P}_N p_1 - \mathcal{P}_N p_2)|^2\, dx \\ \le C_1(1 + \|f_P\|^2_{L^2(\Omega)} + \|f_I\|^2_{L^2(\Omega)} + \|\rho_1^\star s_1^\star\|^2_{L^2(\Omega)} + \|\rho_2^\star s_2^\star\|^2_{L^2(\Omega)}), \end{aligned} \tag{2.27}$$

where C_1 depends on ε and not on N.
Then, up to a subsequence, we have the convergences,

$$p_{1,N} \longrightarrow p_1 \text{ weakly in } H^1_{\Gamma_1}(\Omega),\ \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega \tag{2.28}$$

$$p_{2,N} \longrightarrow p_2 \text{ weakly in } H^1_{\Gamma_1}(\Omega),\ \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{2.29}$$

The convergences in (2.25)-(2.26) with respect to N are obtained in the same manner as for the convergences with respect to n in (2.8) (2.9).

Step 3. Passage to the limit as ε goes to zero. For all $\varepsilon > 0$, we have shown that there exists $(p_{1,\varepsilon}, p_{2,\varepsilon}) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$, satisfying

$$\int_\Omega \phi \frac{\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)-\rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \mathbf{K}M_1^\varepsilon(s_1^\varepsilon)\rho_1(p_1^\varepsilon)\nabla p_1^\varepsilon\cdot\nabla\varphi\,dx$$
$$-\int_\Omega \mathbf{K}M_1(s_1^\varepsilon)\rho_1^2(p_1^\varepsilon)\mathbf{g}\cdot\nabla\varphi\,dx + \eta\int_\Omega \rho_1(p_1^\varepsilon)\nabla(p_1^\varepsilon-p_2^\varepsilon)\cdot\nabla\varphi\,dx$$
$$+\int_\Omega \rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)f_P\varphi\,dx = \int_\Omega \rho_1(p_1^\varepsilon)s_1^I f_I\varphi\,dx, \quad (2.30)$$

$$\int_\Omega \phi \frac{\rho_2(p_2^\varepsilon)Z(s_2^\varepsilon)-\rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \mathbf{K}M_2^\varepsilon(s_2^\varepsilon)\rho_2(p_2^\varepsilon)\nabla p_2^\varepsilon\cdot\nabla\xi\,dx$$
$$-\int_\Omega \mathbf{K}M_2(s_2^\varepsilon)\rho_2^2(p_2^\varepsilon)\mathbf{g}\cdot\nabla\xi\,dx - \eta\int_\Omega \rho_2(p_2^\varepsilon)\nabla(p_1^\varepsilon-p_2^\varepsilon)\cdot\nabla\xi\,dx$$
$$+\int_\Omega \rho_2(p_2^\varepsilon)Z(s_2^\varepsilon)f_P\xi\,dx = \int_\Omega \rho_2(p_2^\varepsilon)s_2^I f_I\xi\,dx, \quad (2.31)$$

for all (φ,ξ) belonging to $H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)$.

We need uniform estimates on the solutions independent of the regularization ε in order to pass to the limit in ε. For that, we are going to use the feature of global pressure. After the passage to the limit in ε, a maximum principle on saturations is possible.
We are now concerned with a uniform estimate on the gradient of $\beta(s_1^\varepsilon)$, and on the global pressure p^ε.

We state the following two lemmas.
Lemma 3.3. *The sequences $(s_i^\varepsilon)_\varepsilon$, $(p^\varepsilon := p_2^\varepsilon + \tilde{p}(s_1^\varepsilon))_\varepsilon$ defined by the proposition 3.1 satisfy*

$$(p^\varepsilon)_\varepsilon \textit{ is uniformly bounded in } H^1_{\Gamma_1}(\Omega) \quad (2.32)$$
$$(\sqrt{\varepsilon}\ \nabla p_i^\varepsilon)_\varepsilon \textit{is uniformly bounded in } L^2(\Omega) \quad (2.33)$$
$$(\beta(s_1^\varepsilon))_\varepsilon \textit{ is uniformly bounded in } H^1(\Omega) \quad (2.34)$$
$$(\nabla f(s_1^\varepsilon))_\varepsilon \textit{ is uniformly bounded in } L^2(\Omega) \quad (2.35)$$

Démonstration. Consider $\varphi = g_1(p_1^\varepsilon) := \int_0^{p_1^\varepsilon}\frac{1}{\rho_1(\zeta)}\,d\zeta \in H^1_{\Gamma_1}(\Omega)$ in (2.30) and $\xi = g_2(p_2^\varepsilon) := \int_0^{p_2^\varepsilon}\frac{1}{\rho_2(\zeta)}\,d\zeta \in H^1_{\Gamma_1}(\Omega)$ in (2.31). Summing these quantities, we obtain

$$
\begin{aligned}
&\int_\Omega \frac{\phi}{h}\Big(\Big(\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)-\rho_1^\star s_1^\star\Big)g_1(p_1^\varepsilon)+\Big(\rho_2(p_2^\varepsilon)Z(s_2^\varepsilon)-\rho_2^\star s_2^\star\Big)g_2(p_2^\varepsilon)\Big)\,dx\\
&+\int_\Omega \mathbf{K}M_1^\varepsilon\nabla p_1^\varepsilon\cdot\nabla p_1^\varepsilon\,dx+\eta\int_\Omega\nabla(p_1^\varepsilon-p_2^\varepsilon)\cdot\nabla(p_1^\varepsilon-p_2^\varepsilon)\,dx\\
&-\int_\Omega \mathbf{K}\rho_1(p_1^\varepsilon)M_1(s_1^\varepsilon)\mathbf{g}\cdot\nabla p_1^\varepsilon\,dx+\int_\Omega \mathbf{K}M_2^\varepsilon\nabla p_2^\varepsilon\cdot\nabla p_2^\varepsilon\,dx\\
&-\int_\Omega \mathbf{K}\rho_2(p_2^\varepsilon)M_2(s_2^\varepsilon)\mathbf{g}\cdot\nabla p_2^\varepsilon\,dx+\int_\Omega\Big(\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)g_1(p_1^\varepsilon)+\rho_2(p_2^\varepsilon)Z(s_2^\varepsilon)g_2(p_2^\varepsilon)\Big)f_P\,dx\\
=&\int_\Omega\Big((\rho_1(p_1^\varepsilon)s_1^Ig_1(p_1^\varepsilon)+\rho_2(p_2^\varepsilon)s_2^Ig_2(p_2^\varepsilon)\Big)f_I\,dx,
\end{aligned}
$$

then,

$$
\begin{aligned}
&\int_\Omega \mathbf{K}M_1\nabla p_1^\varepsilon\cdot\nabla p_1^\varepsilon\,dx+\int_\Omega \mathbf{K}M_2\nabla p_2^\varepsilon\cdot\nabla p_2^\varepsilon\,dx\\
&+\varepsilon\int_\Omega \mathbf{K}\nabla p_1^\varepsilon\cdot\nabla p_1^\varepsilon\,dx+\varepsilon\int_\Omega \mathbf{K}\nabla p_2^\varepsilon\cdot\nabla p_2^\varepsilon\,dx+\eta\int_\Omega\nabla f(s_1)\cdot\nabla f(s_1)\,dx\\
&=\int_\Omega \mathbf{K}\rho_1(p_1^\varepsilon)M_1(s_1^\varepsilon)\mathbf{g}\cdot\nabla p_1^\varepsilon\,dx+\int_\Omega \mathbf{K}\rho_2(p_2^\varepsilon)M_2(s_2^\varepsilon)\mathbf{g}\cdot\nabla p_2^\varepsilon\,dx \qquad (2.36)\\
&-\int_\Omega\Big(\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)g_1(p_1^\varepsilon)+\rho_2(p_2^\varepsilon)Z(s_2^\varepsilon)g_2(p_2^\varepsilon)\Big)f_P\,dx\\
&+\int_\Omega\Big((\rho_1(p_1^\varepsilon)s_1^Ig_1(p_1^\varepsilon)+\rho_2(p_2^\varepsilon)s_2^Ig_2(p_2^\varepsilon)\Big)f_I\,dx\\
-&\int_\Omega \frac{\phi}{h}\Big(\Big(\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)-\rho_1^\star s_1^\star\Big)g_1(p_1^\varepsilon)+\Big(\rho_2(p_2^\varepsilon)Z(s_2^\varepsilon)-\rho_2^\star s_2^\star\Big)g_2(p_2^\varepsilon)\Big)\,dx.
\end{aligned}
$$

The hypothesis (H2) and with the help of Cauchy-Schwarz inequality, we have

$$
\Big|\int_\Omega \mathbf{K}\rho_1(p_1^\varepsilon)M_1(s_1^\varepsilon)\mathbf{g}\cdot\nabla p_1^\varepsilon\,dx\Big|\le C+\frac{k_0}{2}\int_\Omega M_1\nabla p_1^\varepsilon\cdot\nabla p_1^\varepsilon\,dx, \qquad (2.37)
$$

$$
\Big|\int_\Omega \mathbf{K}\rho_2(p_2^\varepsilon)M_2(s_2^\varepsilon)\mathbf{g}\cdot\nabla p_2^\varepsilon\,dx\Big|\le C+\frac{k_0}{2}\int_\Omega M_2\nabla p_2^\varepsilon\cdot\nabla p_2^\varepsilon\,dx, \qquad (2.38)
$$

then the gravity terms in (2.36) on the right hand side are absorbed by pressures dissipative terms. Recall that, the functions $p_i\to g_i(p_i)$ is sub-linear (i.e $|g_i(p_i)|\le\frac{1}{\rho_m}|p_i|$), then from (2.36), one gets

$$
\begin{aligned}
&\int_\Omega M_1^\varepsilon(s_1^\varepsilon)|\nabla p_1^\varepsilon|^2\,dx+\int_\Omega M_2^\varepsilon(s_2^\varepsilon)|\nabla p_2^\varepsilon|^2\,dx+\eta\int_\Omega|\nabla f(s_1)|^2\,dx\\
&\qquad+\varepsilon\int_\Omega|\nabla p_1^\varepsilon|^2\,dx+\varepsilon\int_\Omega|\nabla p_2^\varepsilon|^2\,dx\le C(1+\parallel p_1^\varepsilon\parallel_{L^2(\Omega)}+\parallel p_2^\varepsilon\parallel_{L^2(\Omega)}). \qquad (2.39)
\end{aligned}
$$

Return now to the relationship between pressures and global pressure. From

(4.10), we have $p^\varepsilon = p_2^\varepsilon + \tilde{p}(s_1^\varepsilon) = p_1^\varepsilon + \bar{p}(s_1^\varepsilon)$, and

$$\nabla p^\varepsilon = \nabla p_2^\varepsilon + \frac{M_1(s_1^\varepsilon)}{M(s_1^\varepsilon)}\nabla f(s_1^\varepsilon) = \nabla p_1^\varepsilon - \frac{M_2(s_2^\varepsilon)}{M(s_1^\varepsilon)}\nabla f(s_1^\varepsilon), \tag{2.40}$$

which imply that,

$$\int_\Omega M(s_1^\varepsilon)|\nabla p^\varepsilon|^2\,dx + \int_\Omega \frac{M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)}{M(s_1^\varepsilon)}|\nabla f(s_1^\varepsilon)|^2\,dx = \int_\Omega M_1(s_1^\varepsilon)\nabla p_1^\varepsilon\cdot\nabla p_1^\varepsilon\,dx + \int_\Omega M_2(s_2^\varepsilon)\nabla p_2^\varepsilon\cdot\nabla p_2^\varepsilon\,dx. \tag{2.41}$$

The estimate (2.39) is equivalent to

$$\begin{aligned}
&\int_\Omega M(s_1^\varepsilon)|\nabla p^\varepsilon|^2\,dx + \int_\Omega \frac{M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)}{M(s_1^\varepsilon)}|\nabla f(s_1^\varepsilon)|^2\,dx \\
&+\eta\int_\Omega|\nabla f(s_1)|^2\,dx + \varepsilon\int_\Omega|\nabla p_1^\varepsilon|^2\,dx + \varepsilon\int_\Omega|\nabla p_2^\varepsilon|^2\,dx \\
&\le C(1+\parallel p_1^\varepsilon\parallel_{L^2(\Omega)} + \parallel p_2^\varepsilon\parallel_{L^2(\Omega)}) \\
&\le C(1+\parallel p^\varepsilon\parallel_{L^2(\Omega)} + \parallel \overline{p}(s_1^\varepsilon)\parallel_{L^2(\Omega)} + \parallel \tilde{p}(s_1^\varepsilon)\parallel_{L^2(\Omega)}) \\
&\le C(1+\parallel \nabla p^\varepsilon\parallel_{L^2(\Omega)} + \parallel \overline{p}(s_1^\varepsilon)\parallel_{L^2(\Omega)} + \parallel \tilde{p}(s_1^\varepsilon)\parallel_{L^2(\Omega)}),
\end{aligned}$$

due to the Poincaré's inequality. Finally, using the fact that the function $\tilde{p}$ and $\bar{p}$ are bounded, and the global pressure term on the right hand side in the above inequality can be absorbed by the dissipative term in global pressure, on the left hand side, we deduce that there exists a constant C_1 independent of ε, $C_1 = C_1(h, \rho_m, M_i, \mathbf{g}, f_p, f_I, s_1^I, s_2^I, \rho_1^\star s_1^\star, \rho_2^\star s_2^\star, h, \phi, k_\infty, k_0)$ such that

$$\int_\Omega M(s_1^\varepsilon)|\nabla p^\varepsilon|^2\,dx + \int_\Omega \frac{M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)}{M(s_1^\varepsilon)}|\nabla f(s_1^\varepsilon)|^2\,dx + \eta\int_\Omega|\nabla f(s_1)|^2\,dx + \varepsilon\int_\Omega|\nabla p_1^\varepsilon|^2\,dx + \varepsilon\int_\Omega|\nabla p_2^\varepsilon|^2\,dx \le C_1, \tag{2.42}$$

which establish the estimates (2.32), (2.33) and (2.35). For the estimate (2.34), we use the fact that the second term on the left hand side in (2.42) is bounded and the total mobility is bounded below due to the assumption (H3), we have

$$\int_\Omega M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)|\nabla f(s_1^\varepsilon)|^2\,dx \le m_0 C_1,$$

which implies that,

$$\int_\Omega |\nabla\beta(s_1^\varepsilon)|^2\,dx = \int_\Omega \frac{M_1^2(s_1^\varepsilon)M_2^2(s_2^\varepsilon)}{M^2(s_1^\varepsilon)}|\nabla f(s_1^\varepsilon)|^2\,dx$$
$$\leq \int_\Omega M_1(s_1^\varepsilon)M_2(s_2^\varepsilon)|\nabla f(s_1^\varepsilon)|^2\,dx \leq m_0C_1,$$

and completes the proof of lemma. □

From the previous lemma, we deduce the following convergences.

Lemma 3.4. *(Strong and weak convergences)*
Up to a subsequence the sequence $(s_i^\varepsilon)_\varepsilon$, $(p^\varepsilon)_\varepsilon$, $(p_i^\varepsilon)_\varepsilon$ verify the following convergence

$$p^\varepsilon \longrightarrow p \qquad \textit{weakly in } H^1_{\Gamma_1}(\Omega), \tag{2.43}$$
$$\beta(s_1^\varepsilon) \longrightarrow \beta(s_1) \qquad \textit{weakly in } H^1(\Omega), \tag{2.44}$$
$$p^\varepsilon \longrightarrow p \qquad \textit{almost everywhere in } \Omega \tag{2.45}$$
$$\beta(s_1^\varepsilon) \longrightarrow \beta(s_1) \qquad \textit{almost everywhere in } \Omega \tag{2.46}$$
$$Z(s_1^\varepsilon) \longrightarrow Z(s_1) \qquad \textit{almost everywhere in } \Omega \tag{2.47}$$
$$Z(s_1^\varepsilon) \longrightarrow Z(s_1) \qquad \textit{strongly in } L^2(\Omega) \tag{2.48}$$
$$p_i^\varepsilon \longrightarrow p_i \qquad \textit{almost everywhere in } \Omega. \tag{2.49}$$

Démonstration. The weak convergences (2.43)–(2.44) follows from the uniform estimates (2.32) and (2.34) of lemma 3.3, while

$$p^\varepsilon \longrightarrow p \text{ strongly in } L^2(\Omega) \text{ and a.e. in } \Omega,$$
$$\beta(s_1^\varepsilon) \longrightarrow \beta^\star \text{ strongly in } L^2(\Omega) \text{ and a. e. in } \Omega$$

is due to the compact injection of $H^1_{\Gamma_1}$ in to $L^2(\Omega)$.
As $\beta(s_1) := \beta(Z(s_1))$ and β^{-1} is continuous,

$$Z(s_1^\varepsilon) \longrightarrow Z(s_1) \text{ a. e. in } \Omega,$$

while the Lebesgue theorem ensures the strong convergence (2.48).
The convergence (2.49) is a consequence of (2.45)–(2.47) and the fact that $p_1^\varepsilon := p^\varepsilon - \bar{p}(Z(s_1^\varepsilon))$, $p_2^\varepsilon := p^\varepsilon - \tilde{p}(Z(s_1^\varepsilon))$. □

In order to achieve the proof of Theorem 3.3, it remains to pass to the limit as ε goes to zero in the formulations (2.30)(2.31) and a proof of a maximum principle on saturations.

For all test functions $(\varphi, \xi) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$,

$$\int_\Omega \phi \frac{\rho_1(p_1^\varepsilon)Z(s_1^\varepsilon) - \rho_1^\star s_1^\star}{h}\varphi\,dx + \int_\Omega \mathbf{K} M_1^\varepsilon(s_1^\varepsilon)\rho_1(p_1^\varepsilon)\nabla p_1^\varepsilon \cdot \nabla\varphi\,dx \\ - \int_\Omega \mathbf{K} M_1(s_1^\varepsilon)\rho_1^2(p_1^\varepsilon)\mathbf{g}\cdot\nabla\varphi\,dx + \eta\int_\Omega \rho_1(p_1^\varepsilon)\nabla(p_1^\varepsilon - p_2^\varepsilon)\cdot\nabla\varphi\,dx \\ + \int_\Omega \rho_1(p_1^\varepsilon)Z(s_1^\varepsilon)f_P\varphi\,dx = \int_\Omega \rho_1(p_1^\varepsilon)s_1^I f_I\varphi\,dx,$$

$$\int_\Omega \phi \frac{\rho_2(p_2^\varepsilon)Z(s_2^\varepsilon) - \rho_2^\star s_2^\star}{h}\xi\,dx + \int_\Omega \mathbf{K} M_2^\varepsilon(s_2^\varepsilon)\rho_2(p_2^\varepsilon)\nabla p_2^\varepsilon \cdot \nabla\xi\,dx \\ - \int_\Omega \mathbf{K} M_2(s_2^\varepsilon)\rho_2^2(p_2^\varepsilon)\mathbf{g}\cdot\nabla\xi\,dx - \eta\int_\Omega \rho_2(p_2^\varepsilon)\nabla(p_1^\varepsilon - p_2^\varepsilon)\cdot\nabla\xi\,dx \\ + \int_\Omega \rho_2(p_2^\varepsilon)Z(s_2^\varepsilon)f_P\xi\,dx = \int_\Omega \rho_2(p_2^\varepsilon)s_2^I f_I\xi\,dx,$$

The first terms of the above equalities converge due to the strong convergence of $\rho_i(p_i^\varepsilon)Z(s_i^\varepsilon)$ to $\rho_i(p_i)Z(s_i)$ in $L^2(\Omega)$.
The second terms can be written as,

$$\int_\Omega \mathbf{K} M_i^\varepsilon(s_i^\varepsilon)\rho_i(p_i^\varepsilon)\nabla p_i^\varepsilon\cdot\nabla\varphi\,dx = \int_\Omega \mathbf{K} M_i(s_i^\varepsilon)\rho_i(p_i^\varepsilon)\nabla p^\varepsilon\cdot\nabla\varphi\,dx \\ + \int_\Omega \mathbf{K}\rho_i(p_i^\varepsilon)\nabla\beta(s_i^\varepsilon)\cdot\nabla\varphi\,dx + \sqrt{\varepsilon}\int_\Omega \mathbf{K}\rho_i(p_i^\varepsilon)(\sqrt{\varepsilon}\;\nabla p_i^\varepsilon)\cdot\nabla\varphi\,dx. \quad (2.50)$$

The first two terms on the right hand side of the equation converge arguing in two steps. Firstly, the Lebsgue theorem and the convergences (2.47)(2.49) establish

$$\rho_i(p_i^\varepsilon)M_i(s_i^\varepsilon)\nabla\varphi \longrightarrow \rho_i(p_i)M_i(s_i)\nabla\varphi \text{ strongly in } (L^2(Q_T))^d,$$

$$\rho_i(p_i^\varepsilon)\nabla\varphi \longrightarrow \rho_i(p_i)\nabla\varphi \text{ strongly in } (L^2(Q_T))^d.$$

Secondly, the weak convergence on pressure (2.43) combined to the above strong convergence validate the convergence for the first term of the right hand side of (2.50), and the weak convergence (2.44) combined to the above strong convergence validate the convergence for the second term of the right hand side of (2.50).
The third term converges to zero due to the uniform estimate (2.33), and this achieves the passage to the limit on the second terms.
The convergences of the fourth terms of the above equations are due to the uniform estimate (2.35). The other terms converge using (2.47)(2.49) and the Lebesgue dominated convergence theorem.

In summarize, we have shown, there exists $(p_1^h, p_1^h) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ solution of

$$\begin{aligned}&\int_\Omega \phi\frac{\rho_1(p_1^h)Z(s_1^h)-\rho_1^\star s_1^\star}{h}\varphi\,dx+\int_\Omega \mathbf{K}M_1(s_1^h)\rho_1(p_1^h)\nabla p_1^h\cdot\nabla\varphi\,dx\\&-\int_\Omega \mathbf{K}M_1(s_1^h)\rho_1^2(p_1^h)\mathbf{g}\cdot\nabla\varphi\,dx+\eta\int_\Omega\rho_1(p_1^h)\nabla(p_1^h-p_2^h)\cdot\nabla\varphi\,dx\\&+\int_\Omega\rho_1(p_1^h)Z(s_1^h)f_P\varphi\,dx=\int_\Omega\rho_1(p_1^h)s_1^If_I\varphi\,dx,\end{aligned}\tag{2.51}$$

$$\begin{aligned}&\int_\Omega \phi\frac{\rho_2(p_2^h)Z(s_2^h)-\rho_2^\star s_2^\star}{h}\xi\,dx+\int_\Omega \mathbf{K}M_2(s_2^h)\rho_2(p_2^h)\nabla p_2^h\cdot\nabla\xi\,dx\\&-\int_\Omega \mathbf{K}M_2(s_2^h)\rho_2^2(p_2^h)\mathbf{g}\cdot\nabla\xi\,dx-\eta\int_\Omega\rho_2(p_2^h)\nabla(p_1^h-p_2^h)\cdot\nabla\xi\,dx\\&+\int_\Omega\rho_2(p_2^h)Z(s_2^h)f_P\xi\,dx=\int_\Omega\rho_2(p_2^h)s_2^If_I\xi\,dx,\end{aligned}\tag{2.52}$$

for all $\varphi,\xi\in H^1_{\Gamma_1}(\Omega)$,
Lemma 3.5. *(***Maximum principle***) Under the conditions of Theorem 3.3, the saturation functions s_1^h and s_2^h which verify (2.51)-(2.52) are between zero and one a.e in Ω.*

Démonstration. It is enough to show that $s_i^h\geq 0$ a.e in Ω. For that, consider $\varphi=-(s_1)^-,\xi=-(s_2)^-$ respectively in (2.51) and (2.52) and by taking into consideration the definition of the map Z, and according to the extension of the mobility of each phase, $M_i(s_i^h)(s_i^h)^-=0\quad(i=1,2.)$ we get

$$\int_\Omega\phi\frac{\rho_1^\star s_1^\star}{h}(s_1^h)^-\,dx+\eta\int_\Omega\bar f'(s_1^h)\nabla(s_1^h)^-\cdot\nabla(s_1^h)^-\,dx=-\int_\Omega\rho_1(p_1^h)s_1^If_I(s_1^h)^-\,dx,$$

and

$$\int_\Omega\phi\frac{\rho_2^\star s_2^\star}{h}(s_2^h)^-\,dx+\eta\int_\Omega\bar f'(s_1^h)\nabla(s_2^h)^-\cdot\nabla(s_2^h)^-\,dx=-\int_\Omega\rho_2(p_2^h)s_2^If_I(s_2^h)^-\,dx.$$

Since it is possible to choose an extension $\bar f$ of f outside $[0,1]$ in a way that ensures $\bar f'(s_1)$ different from zero outside $[0,1]$, we get

$$\eta\int_\Omega|\nabla(s_i^h)^-|^2\,dx\leq 0\quad(i=1,2.),$$

which proves the maximum principle since s_i^- vanishes on Γ_1, $i=1,2$. □

After this maximum principle, the weak formulations (1.27) and (1.28) are established, and thus the theorem 3.3 is then established.

3 Proof of Theorem 3.2

The proof is based on a semi-discretization method in time [3]. Let be $T > 0$, $N \in \mathbb{N}^*$ and $h = \frac{T}{N}$. We define the following sequence parameterized by h :

$$p_{i,h}^0(x) = p_i^0(x) \text{ a.e. in } \Omega \qquad i = 1, 2, \tag{3.1}$$

for all $n \in [0, N-1]$, consider $(p_{1,h}^n, p_{2,h}^n) \in L^2(\Omega) \times L^2(\Omega)$ with $\rho_1(p_{i,h}^n)s_{i,h}^n \geq 0$ for $i = 1, 2$, denote by $(f_P)_h^{n+1} = \dfrac{1}{h}\displaystyle\int_{nh}^{(n+1)h} f_P(\tau)\, d\tau$, $(f_I)_h^{n+1} = \dfrac{1}{h}\displaystyle\int_{nh}^{(n+1)h} f_I(\tau)\, d\tau$ and $(s_i^I)_h^{n+1} = \dfrac{1}{h}\displaystyle\int_{nh}^{(n+1)h} s_i^I(\tau)\, d\tau$ for $i = 1, 2$, then define $(p_{1,h}^{n+1}, p_{2,h}^{n+1})$ solution of

$$\begin{aligned}\phi\frac{\rho_1(p_{1,h}^{n+1})s_{1,h}^{n+1} - \rho_1(p_{1,h}^n)s_{1,h}^n}{h} &- \operatorname{div}(\mathbf{K}M_1(s_{1,h}^{n+1})\rho_1(p_{1,h}^{n+1})\nabla p_{1,h}^{n+1})\\ &+ \operatorname{div}(\mathbf{K}\rho_1^2(p_{1,h}^{n+1})M_1(s_{1,h}^{n+1})\mathbf{g}) - \eta\operatorname{div}(\rho_1(p_{1,h}^{n+1})\nabla(p_{1,h}^{n+1} - p_{2,h}^{n+1}))\\ &+ \rho_1(p_{1,h}^{n+1})s_{1,h}^{n+1}(f_P)_h^{n+1} = \rho_1(p_{1,h}^{n+1})(s_1^I)_h^{n+1}(f_I)_h^{n+1},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\phi\frac{\rho_2(p_{2,h}^{n+1})s_{2,h}^{n+1} - \rho_2(p_{2,h}^n)s_{2,h}^n}{h} &- \operatorname{div}(\mathbf{K}M_2(s_{2,h}^{n+1})\rho_2(p_{2,h}^{n+1})\nabla p_{2,h}^{n+1})\\ &+ \operatorname{div}(\mathbf{K}\rho_2^2(p_{2,h}^{n+1})M_2(s_{2,h}^{n+1})\mathbf{g}) + \eta\operatorname{div}(\rho_2(p_{2,h}^{n+1})\nabla(p_{1,h}^{n+1} - p_{2,h}^{n+1}))\\ &+ \rho_2(p_{2,h}^{n+1})s_{2,h}^{n+1}(f_P)_h^{n+1} = \rho_2(p_{2,h}^{n+1})(s_2^I)_h^{n+1}(f_I)_h^{n+1},\end{aligned} \tag{3.3}$$

with the boundary conditions (1.24). This sequence is well defined for all $n \in [0, N-1]$ by virtue of theorem 3.3. As a matter of fact, for given $s_{i,h}^n\rho_i(p_{i,h}^n) \geq 0$ and $\rho_i(p_{i,h}^n)s_{i,h}^n \in L^2(\Omega)$, $i = 1, 2$, we construct $(p_{1,h}^{n+1}, p_{2,h}^{n+1}) \in H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega)$ so that $s_{i,h}^{n+1} \in [0, 1]$.

Now, we are concerned with uniform estimates with respect to h. We state the following lemma.

Lemma 3.6. *(Uniform estimates with respect to h) The solutions of (3.2)-(3.3) satisfy*

$$\begin{aligned}&\frac{1}{h}\int_\Omega \phi\Big(\mathcal{H}_1(p_{1,h}^{n+1})s_{1,h}^{n+1} - \mathcal{H}_1(p_{1,h}^n)s_{1,h}^n\Big)\, dx\\ +&\frac{1}{h}\int_\Omega \phi\Big(\mathcal{H}_2(p_{2,h}^{n+1})s_{2,h}^{n+1} - \mathcal{H}_2(p_{2,h}^n)s_{2,h}^n\Big)\, dx\\ +&\frac{1}{h}\int_\Omega \phi\Big(\mathcal{F}(s_{1,h}^{n+1}) - \mathcal{F}(s_{1,h}^n)\Big)\, dx + \eta\int_\Omega |\nabla(p_{1,h}^{n+1} - p_{2,h}^{n+1})|^2\, dx\\ +&k_0\int_\Omega M_1(s_{1,h}^{n+1})\nabla p_{1,h}^{n+1}\cdot\nabla p_{1,h}^{n+1}\, dx + k_0\int_\Omega M_2(s_{2,h}^{n+1})\nabla p_{2,h}^{n+1}\cdot\nabla p_{2,h}^{n+1}\, dx\\ \leq& C(1 + \|(f_P)_h^{n+1}\|^2_{L^2(\Omega)} + \|(f_I)_h^{n+1}\|^2_{L^2(\Omega)})\end{aligned} \tag{3.4}$$

where C does not depend on h, and for $i = 1, 2$

$$\mathcal{H}_i(p_i) := \rho_i(p_i) g_i(p_i) - p_i, \quad \mathcal{F}(s) := \int_0^s f(\zeta)\, d\zeta \ \textit{and}\ g_i(p_i) = \int_0^{p_i} \frac{1}{\rho_i(\zeta)}\, d\zeta.$$

Démonstration. First of all, let us prove that : for all $s_i \geq 0$ and $s_i^\star \geq 0$ such that $s_1 + s_2 = s_1^\star + s_2^\star = 1$,

$$\begin{aligned}\Big(\rho_1(p_1)s_1 - \rho_1(p_1^\star)s_1^\star\Big) g_1(p_1) + \Big(\rho_2(p_2)s_2 - \rho_2(p_2^\star)s_2^\star\Big) g_2(p_2) \\ \geq \mathcal{H}_1(p_1)s_1 - \mathcal{H}_1(p_1^\star)s_1^\star + \mathcal{H}_2(p_2)s_2 - \mathcal{H}_2(p_2^\star)s_2^\star + \mathcal{F}(s_1) - \mathcal{F}(s_1^\star). \end{aligned} \tag{3.5}$$

Let us denote by $\mathcal{J}$ the left hand side of (3.5),

$$\mathcal{J} = \Big(\rho_1(p_1)s_1 - \rho_1(p_1^\star)s_1^\star\Big) g_1(p_1) + \Big(\rho_2(p_2)s_2 - \rho_2(p_2^\star)s_2^\star\Big) g_2(p_2).$$

Since the function g_i is concave, we have

$$g_i(p_i) \leq g_i(p_i^\star) + g_i'(p_i^\star)(p_i - p_i^\star) = g_i(p_i^\star) + \frac{1}{\rho_i(p_i^\star)}(p_i - p_i^\star). \tag{3.6}$$

From the definition of $\mathcal{H}_i$, we have

$$\begin{aligned}\mathcal{J} = & \Big[\Big(\rho_1(p_1)s_1 g_1(p_1) - s_1 p_1\Big) + s_1 p_1 - \rho_1(p_1^\star)s_1^\star g_1(p_1)\Big] \\ & + \Big[\Big(\rho_2(p_2)s_2 g_2(p_2) - s_2 p_2\Big) + s_2 p_2 - \rho_2(p_2^\star)s_2^\star g_2(p_2)\Big] \\ = & \ s_1\mathcal{H}_1(p_1) + s_1 p_1 - \rho_1(p_1^\star)s_1^\star g_1(p_1) + s_2\mathcal{H}_2(p_2) + s_2 p_2 - \rho_2(p_2^\star)s_2^\star g_2(p_2)\end{aligned}$$

and the concavity property of g_i leads to

$$\begin{aligned}\mathcal{J} & \geq s_1\mathcal{H}_1(p_1) - s_1^\star\mathcal{H}_1(p_1^\star) + s_2\mathcal{H}_2(p_2) - s_2^\star\mathcal{H}_2(p_2^\star) + s_1p_1 - s_1^\star p_1 + s_2 p_2 - s_2^\star p_2 \\ & \geq s_1\mathcal{H}_1(p_1) - s_1^\star\mathcal{H}_1(p_1^\star) + s_2\mathcal{H}_2(p_2) - s_2^\star\mathcal{H}_2(p_2^\star) + s_1\Big(p_1 - p_2\Big) - s_1^\star\Big(p_1 - p_2\Big) \\ & = s_1\mathcal{H}_1(p_1) - s_1^\star\mathcal{H}_1(p_1^\star) + s_2\mathcal{H}_2(p_2) - s_2^\star\mathcal{H}_2(p_2^\star) + \Big(s_1 - s_1^\star\Big) f(s_1). \end{aligned} \tag{3.7}$$

Since the function $\mathcal{F}$ is convex, then

$$(s_1 - s_1^\star) f(s_1) \geq \mathcal{F}(s_1) - \mathcal{F}(s_1^\star). \tag{3.8}$$

The above inequalities (3.7) and (3.8) ensure that the assertion (3.5) is satisfied. Let us multiply scalarly (3.2) with $g_1(p_{1,h}^{n+1})$ and add the scalar product of (3.3) with $g_2(p_{2,h}^{n+1})$, we have

$$\frac{1}{h}\int_\Omega \phi\Big(\Big(\rho_1(p_{1,h}^{n+1})s_{1,h}^{n+1} - \rho_1(p_{1,h}^{n})s_{1,h}^{n}\Big)g_1(p_{1,h}^{n+1})$$
$$+\Big(\rho_2(p_{2,h}^{n+1})s_{2,h}^{n+1} - \rho_2(p_{2,h}^{n})s_{2,h}^{n}\Big)g_2(p_{2,h}^{n+1})\Big)\,dx$$
$$+\int_\Omega \mathbf{K}M_1(s_{1,h}^{n+1})\nabla p_{1,h}^{n+1}\cdot\nabla p_{1,h}^{n+1}\,dx + \int_\Omega \mathbf{K}M_2(s_{2,h}^{n+1})\nabla p_{2,h}^{n+1}\cdot\nabla p_{2,h}^{n+1}\,dx$$
$$+\eta\int_\Omega |\nabla f(s_{1,h}^{n+1})|^2\,dx = \int_\Omega \mathbf{K}M_1(s_{1,h}^{n+1})\rho_1(p_{1,h}^{n+1})\mathbf{g}\cdot\nabla p_{1,h}^{n+1}\,dx$$
$$+\int_\Omega \mathbf{K}M_2(s_{2,h}^{n+1})\rho_2(p_{2,h}^{n+1})\mathbf{g}\cdot\nabla p_{1,h}^{n+1}\,dx - \int_\Omega \rho_1(p_{1,h}^{n+1})s_{1,h}^{n+1}(f_P)_h^{n+1}g_1(p_{1,h}^{n+1})\,dx$$
$$-\int_\Omega \rho_2(p_{2,h}^{n+1})s_{2,h}^{n+1}(f_P)_h^{n+1}g_2(p_{2,h}^{n+1})\,dx + \int_\Omega \rho_1(p_{1,h}^{n+1})(s_1^I)_h^{n+1}(f_I)_h^{n+1}g_1(p_{1,h}^{n+1})\,dx$$
$$+\int_\Omega \rho_2(p_{2,h}^{n+1})(s_2^I)_h^{n+1}(f_I)_h^{n+1}g_2(p_{2,h}^{n+1})\,dx. \tag{3.9}$$

Using (3.5) and following the demonstration as *lemma* 3.3 one gets (3.4). □

For a given sequence $(u_h^n)_n$, let us denote

$$u_h(0) = u_h^0,$$
$$u_h(t) = \sum_{n=0}^{N-1} u_h^{n+1}\chi_{]nh,(n+1)h]}(t), \quad \forall t\in]0,T] \tag{3.10}$$

and

$$\tilde{u}_h(t) = \sum_{n=0}^{N-1}\Big((1+n-\frac{t}{h})u_h^n + (\frac{t}{h}-n)u_h^{n+1}\Big)\chi_{[nh,(n+1)h]}(t), \quad \forall t\in[0,T]. \tag{3.11}$$

Then,

$$\partial_t\tilde{u}_h(t) = \frac{1}{h}\sum_{n=0}^{N-1}((u_h^{n+1}-u_h^n)\chi_{]nh,(n+1)h[}(t), \quad \forall t\in[0,T]\backslash\{\cup_{n=0}^N nh\}$$

Let the functions $p_{i,h}$ and $s_{i,h}$ be defined as in (3.10). For $i=1,2$, we denote by $r_{i,h}$ the function defined by (3.10) corresponding to $r_{i,h}^n = \rho_i(p_{i,h}^n)s_{i,h}^n$ and $\tilde{r}_{i,h}$ the function defined by (3.11) corresponding to $r_{i,h}^n$. In the same way, we denote by $f_{P,h}$, $f_{I,h}$ and $(s_i^I)_h$ the functions corresponding to $(f_P)_h^{n+1}$, $(f_I)_h^{n+1}$ and $(s_i^I)_h^{n+1}$ respectively.

Proposition 3.2. *The sequence*

$$(s_{2,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_1}(\Omega)), \tag{3.12}$$
$$(p_{i,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1_{\Gamma_1}(\Omega)), i=1,2 \tag{3.13}$$
$$(r_{i,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega)), i=1,2 \tag{3.14}$$
$$(\tilde{r}_{i,h})_h \text{ is uniformly bounded in } L^2(0,T;H^1(\Omega)), i=1,2 \tag{3.15}$$
$$(\phi\partial_t\tilde{r}_{i,h})_h \text{ is uniformly bounded in } L^2(0,T;(H^1_{\Gamma_1}(\Omega))'), i=1,2. \tag{3.16}$$

Démonstration. At the beginning of this proof, we indicate to some useful remarks which can be established by a classical calculations,

$$\int_{Q_T} M_i(s_{i,h})|\nabla p_{i,h}|^2\,dxdt = h\sum_{n=0}^{N-1}\int_\Omega M_i(s_{i,h}^{n+1})|\nabla p_{i,h}^{n+1}|^2\,dx \quad (i=1,2.), \tag{3.17}$$

$$\int_{Q_T} |\nabla f(s_{1,h})|^2\,dxdt = h\sum_{n=0}^{N-1}\int_\Omega |\nabla f(s_{1,h}^{n+1})|^2\,dx, \tag{3.18}$$

$$\int_{Q_T} |f_p(t,x)|^2\,dtdx \geq h\sum_{n=0}^{N-1}\|(f_p)_h^{n+1}\|_{L^2(\Omega)}, \tag{3.19}$$

$$\int_{Q_T} |f_I(t,x)|^2\,dtdx \geq h\sum_{n=0}^{N-1}\|(f_I)_h^{n+1}\|_{L^2(\Omega)}. \tag{3.20}$$

Now, multiply (3.4) by h and summing it from $n=0$ to $n=N-1$,

$$\begin{aligned}
&\int_\Omega \phi\mathcal{H}_1(p_{1,h}(T))s_{1,h}(T) + \phi\mathcal{H}_2(p_{2,h}(T))s_{2,h}(T)\,dx\\
&+k_0\int_{Q_T} M_1(s_{1,h})|\nabla p_{1,h}|^2\,dxdt + k_0\int_{Q_T} M_2(s_{2,h})|\nabla p_{2,h}|^2\,dxdt\\
&+\eta\int_{Q_T}|\nabla f(s_{1,h})|^2\,dxdt \leq \int_\Omega\Big(\phi\mathcal{H}_1(p_{1,h}(0))s_{1,h}(0) + \phi\mathcal{H}_2(p_{2,h}(0))s_{2,h}(0)\Big)dx\\
&+\mathcal{F}(s_{1,h}(0)) - \mathcal{F}(s_{1,h}(T)) + C\Big(1+\|f_P\|^2_{L^2(Q_T)} + \|f_I\|^2_{L^2(Q_T)}\Big),
\end{aligned} \tag{3.21}$$

where C is a constant independent of h.
The positivity of the first term on the left hand side of (3.21) ensures that there exists a positive constant C independent of h such that

$$\begin{aligned}
k_0\int_{Q_T} M_1(s_{1,h})|\nabla p_{1,h}|^2\,dxdt + k_0\int_{Q_T} M_2(s_{2,h})|\nabla p_{2,h}|^2\,dxdt&\\
+\eta\int_{Q_T}|\nabla f(s_{1,h})|^2\,dxdt \leq C&,
\end{aligned}$$

since we have,

$$\int_{Q_T} M_1(s_{1,h})|\nabla p_{1,h}|^2\,dxdt + \int_{Q_T} M_2(s_{2,h})|\nabla p_{2,h}|^2\,dxdt$$
$$= \int_{Q_T} M(s_{1,h})|\nabla p_h|^2\,dxdt + \int_{Q_T} \frac{M_1(s_{1,h})M_2(s_{2,h})}{M(s_{1,h})}|\nabla f(s_{1,h})|^2\,dxdt, \quad (3.22)$$

then, we deduce

$$\int_{Q_T} M(s_{1,h})|\nabla p_h|^2\,dxdt + \eta\int_{Q_T} |\nabla f(s_{1,h})|^2\,dxdt \le C. \quad (3.23)$$

For the first estimate (3.12) and first of all, let us indicate to the fact that,

$$p_{1,h}(t,x) - p_{2,h}(t,x) = 0 = f(s_{1,h}(t,x)) \qquad \text{for } \ x \in \Gamma_1$$

which gives that $s_{2,h|\Gamma_1} = 0$. The assumption (H6) on the capillary function f with the second term on the right hand side of (3.23) lead to

$$\int_{Q_T} |\nabla s_{1,h}|^2\,dxdt \le C,$$

where C is a constant independent of h, which establishes (3.12).

Since we have,

$$\nabla p_{1,h} = \nabla p_h + \frac{M_2}{M}\nabla f(s_{1,h}) \ \text{ and } \quad \nabla p_{2,h} = \nabla p_h - \frac{M_1}{M}\nabla f(s_{1,h}),$$

then, the estimate (3.13) becomes a consequence of (3.23).
The uniform estimate (3.14) is a consequence of the two previous ones since the densities ρ_i are bounded and of class $\mathcal{C}^1$functions as well as the saturations $0 \le s_{i,h} \le 1$,

$$\nabla r_{i,h} = \sum_{n=0}^{N-1}\Big(\rho_i'(p_{i,h}^{n+1})s_{i,h}^{n+1}\nabla p_{i,h}^{n+1} + \rho_i(p_{i,h}^{n+1})\nabla s_{i,h}^{n+1}\Big)\chi_{]nh,(n+1)h]}(t).$$

Now, for estimate (3.15) we have,

$$\nabla \tilde{r}_{i,h} = \sum_{n=0}^{N-1}\Big((1+n-\frac{t}{h})[\rho_i'(p_{i,h}^{n})s_{i,h}^{n}\nabla p_{i,h}^{n} + \rho_i(p_{i,h}^{n})\nabla s_{i,h}^{n}]$$
$$+ (\frac{t}{h}-n)[\rho_i'(p_{i,h}^{n+1})s_{i,h}^{n+1}\nabla p_{i,h}^{n+1} + \rho_i(p_{i,h}^{n+1})\nabla s_{i,h}^{n+1}]\Big)\chi_{]nh,(n+1)h]}(t). \quad (3.24)$$

since the densities ρ_i are bounded and of class $\mathcal{C}^1$ functions as well as the saturations $0 \leq s_{i,h}^n \leq 1$,

$$|\nabla \tilde{r}_{i,h}|^2 \leq C \sum_{n=0}^{N-1} \Big(|\nabla p_{i,h}^n|^2 + |\nabla s_{i,h}^n|^2 + |\nabla p_{i,h}^{n+1}|^2 + |\nabla s_{i,h}^{n+1}|^2 \Big) \chi_{]nh,(n+1)h]}(t), \quad (3.25)$$

and this implies that,

$$||\nabla \tilde{r}_{i,h}||_{L^2(Q_T)}^2 \leq C(\|\nabla p_{i,h}^0\|_{L^2(\Omega)}^2 + \|\nabla s_{i,h}^0\|_{L^2(\Omega)}^2 + \|\nabla p_{i,h}\|_{L^2(Q_T)}^2 + \|\nabla s_{i,h}\|_{L^2(Q_T)}^2), \quad (3.26)$$

where C is a constant independent of h, and the estimate (3.15) is established. From equations (3.2) and (3.3), we have for all $\varphi \in L^2(0,T; H^1_{\Gamma_1}(\Omega))$,

$$\begin{aligned} \langle \phi \partial_t \tilde{r}_{i,h}, \varphi \rangle &= -\int_{Q_T} \mathbf{K} M_i(s_{i,h}) \rho_i(p_{i,h}) \nabla p_{i,h} \cdot \nabla \varphi \, dxdt \\ &+ \int_{Q_T} \mathbf{K} \rho_i^2(p_{i,h}) M_i(s_{i,h}) \mathbf{g} \cdot \nabla \varphi \, dxdt + \eta(-1)^i \int_{Q_T} \nabla(p_{1,h} - p_{2,h}) \cdot \nabla \varphi \, dxdt \\ &- \int_{Q_T} \rho_i(p_{i,h}) s_{i,h} f_{P,h} \varphi \, dxdt + \int_{Q_T} \rho_i(p_h) s_{i,h}^I f_{I,h} \varphi \, dxdt. \end{aligned}$$

The above estimates (3.12)–(3.13) with (3.23) ensure that $(\phi \partial_t \tilde{r}_{i,h})_h$ is uniformly bounded in $L^2(0,T; (H^1_{\Gamma_1}(\Omega))')$. □

The next step is to pass from an elliptic problem to a parabolic one. Then, we pass to the limit on h, using some compactness theorems.

Proposition 3.3. *(Convergence with respect to h) We have the following convergences as h goes to zero,*

$$\|r_{i,h} - \tilde{r}_{i,h}\|_{L^2(Q_T)} \longrightarrow 0, \quad (3.27)$$
$$s_{2,h} \longrightarrow s_2 \text{ weakly in } L^2(0,T; H^1_{\Gamma_1}(\Omega)), \quad (3.28)$$
$$p_{i,h} \longrightarrow p_i \text{ weakly in } L^2(0,T; H^1_{\Gamma_1}(\Omega)), \quad (3.29)$$
$$r_{i,h} \longrightarrow r_i \text{ strongly in } L^2(Q_T). \quad (3.30)$$

Furthermore,

$$s_{i,h} \longrightarrow s_i \text{ almost everywhere in } Q_T, \quad (3.31)$$
$$0 \leq s_i \leq 1 \text{ almost everywhere in } Q_T, \quad (3.32)$$
$$p_{i,h} \longrightarrow p_i \text{ almost everywhere in } Q_T, \quad (3.33)$$

and

$$r_i = \rho_i(p_i)s_i \text{ almost everywhere in } Q_T. \tag{3.34}$$

Finally, we have,

$$\phi\partial_t\tilde{r}_{i,h} \longrightarrow \phi\partial_t(\rho_i(p_i)s_i) \text{ weakly in } L^2(0,T;(H^1_{\Gamma_1}(\Omega))'). \tag{3.35}$$

Démonstration. Note that

$$\begin{aligned}\|r_{i,h}-\tilde{r}_{i,h}\|^2_{L^2(Q_T)} &= \sum_{n=0}^{N-1}\int_{nh}^{(n+1)h}\|((1+n-\frac{t}{h})(r^{n+1}_{i,h}-r^n_{i,h})\|^2_{L^2(\Omega)}\,dt\\ &= \frac{h}{3}\sum_{n=0}^{N-1}\|r^{n+1}_{i,h}-r^n_{i,h}\|^2_{L^2(\Omega)}.\end{aligned}$$

We multiply scalarly (3.2) and (3.3) respectively with $r^{n+1}_{1,h}-r^n_{1,h}$ and $r^{n+1}_{2,h}-r^n_{2,h}$. Then, summing for $n=0$ to $N-1$, we get
for $i=2$,

$$\begin{aligned}&\frac{\phi_0}{h}\sum_{n=0}^{N-1}\|r^{n+1}_{2,h}-r^n_{2,h}\|^2_{L^2(\Omega)} \le\\ &\quad C\sum_{n=0}^{N-1}\Big(\|\nabla r^n_{2,h}\|^2_{L^2(\Omega)}+\|\nabla r^{n+1}_{2,h}\|^2_{L^2(\Omega)}+\|\nabla s^{n+1}_{2,h}\|^2_{L^2(\Omega)}+\|\nabla p^{n+1}_{2,h}\|^2_{L^2(\Omega)}\\ &\qquad\qquad+\|(f_P)^{n+1}_h\|^2_{L^2(\Omega)}+\|(f_I)^{n+1}_h\|^2_{L^2(\Omega)}\Big).\end{aligned}$$

This yields to

$$\begin{aligned}\sum_{n=0}^{N-1}\|r^{n+1}_{2,h}-r^n_{2,h}\|^2_{L^2(\Omega)} \le C\Big(1+\|\nabla r_{2,h}\|^2+\|\nabla s_{2,h}\|^2_{L^2(Q_T)}+\|\nabla p_{2,h}\|^2_{L^2(Q_T)}\\ +\|f_P\|^2_{L^2(Q_T)}+\|f_I\|^2_{L^2(Q_T)}\Big).\end{aligned}$$

And from (3.12),(3.13), and (3.14), we conclude that

$$\|r_{2,h}-\tilde{r}_{2,h}\|_{L^2(Q_T)} \longrightarrow 0,$$

for $i = 1$,

$$\frac{\phi_0}{h}\sum_{n=0}^{N-1}\|r_{1,h}^{n+1}-r_{1,h}^{n}\|^2_{L^2(\Omega)} \leq$$

$$C\sum_{n=0}^{N-1}\Big(\|\nabla r_{1,h}^{n}\|^2_{L^2(\Omega)}+\|\nabla r_{1,h}^{n+1}\|^2_{L^2(\Omega)}+\Big|\int_{\Gamma_1}\mathbf{K}\rho_1^2(p_{1,h}^{n+1})M_1(s_{1,h}^{n+1})\mathbf{g}.\nu(r_{1,h}^{n+1}-r_{1,h}^{n})\,d\gamma\Big|$$
$$+\|\nabla s_{2,h}^{n+1}\|^2_{L^2(\Omega)}+\|\nabla p_{1,h}^{n+1}\|^2_{L^2(\Omega)}+\|(f_P)_h^{n+1}\|^2_{L^2(\Omega)}+\|(f_I)_h^{n+1}\|^2_{L^2(\Omega)}\Big).$$

where ν is the outward normal to the injection boundary.
This yields with the help of trace theory to

$$\sum_{n=0}^{N-1}\|r_{1,h}^{n+1}-r_{1,h}^{n}\|^2_{L^2(\Omega)} \leq C\Big(1+\|\nabla r_{1,h}\|^2+\|\nabla s_{2,h}\|^2_{L^2(Q_T)}+\|\nabla p_{1,h}\|^2_{L^2(Q_T)}$$
$$+\|f_P\|^2_{L^2(Q_T)}+\|f_I\|^2_{L^2(Q_T)}\Big).$$

And from (3.12),(3.13), and (3.14), we conclude that

$$\|r_{1,h}-\tilde{r}_{1,h}\|_{L^2(Q_T)} \longrightarrow 0,$$

and this achieve (3.27).
From (3.13) (3.12), the sequences $(p_{i,h})_h$, $(s_{2,h})_h$ are uniformly bounded in $L^2(0,T;H^1_{\Gamma_1}(\Omega))$, we have up to a subsequence the convergence results (3.28), (3.29).
The sequences $(\tilde{r}_{i,h})_h$ are uniformly bounded in $L^2(0,T;H^1(\Omega))$. In light of (3.16) we have the strong convergence

$$\tilde{r}_{i,h} \longrightarrow r_i \text{ strongly in } L^2(Q_T). \tag{3.36}$$

This compactness result is classical and can be found in [62], [22] when the porosity is constant, and under the assumption (H1) (the porosity belongs to $W^{1,\infty}(\Omega)$), the proof can be adapted with minor modifications.
The convergence (3.36) with (3.27) ensures the following strong convergences

$$\rho_1(p_{1,h})s_{1,h} \longrightarrow r_1 \text{ strongly in } L^2(Q_T) \text{ and a.e in } Q_T, \tag{3.37}$$
$$\rho_2(p_{2,h})s_{2,h} \longrightarrow r_2 \text{ strongly in } L^2(Q_T) \text{ and a.e in } Q_T, \tag{3.38}$$

and this achieve (3.30)

We are now concerned with almost everywhere convergence on pressures $p_{i,h}$ and saturations $s_{i,h}$.

Denote by

$$u = \rho_1(p_{1,h})s_{1,h}, \;\; v = \rho_2(p_{1,h} - f(s_{1,h}))(1 - s_{1,h}).$$

Define the map $H : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+ \times [0,1]$ defined by

$$H(u, v) = (p_{1,h}, s_{1,h}) \tag{3.39}$$

where u and v are solutions of the system

$$u(p_{1,h}, s_{1,h}) = \rho_1(p_{1,h})s_{1,h}.$$

$$v(p_{1,h}, s_{1,h}) = \rho_2(p_{1,h} - f(s_{1,h}))(1 - s_{1,h}).$$

Note that H is well defined as a diffeomorphism,

$$\begin{aligned}
\begin{vmatrix} \dfrac{\partial u}{\partial p_{1,h}} & \dfrac{\partial u}{\partial s_{1,h}} \\ \dfrac{\partial v}{\partial p_{1,h}} & \dfrac{\partial v}{\partial s_{1,h}} \end{vmatrix} =& - \rho_1'(p_{1,h})\rho_2(p_{1,h} - f(s_{1,h}))s_{1,h} \\
& - \rho_1(p_{1,h})\rho_2'(p_{1,h} - f(s_{1,h}))(1 - s_{1,h}) \\
& - \rho_1'(p_{1,h})s_{1,h}(1 - s_{1,h})\rho_2'(p_{1,h} - f(s_{1,h}))f'(s_{1,h}) < 0.
\end{aligned}$$

As we have the almost everywhere convergences (3.37),(3.38) and the map H defined in (4.16) is continues, we deduce

$$p_{1,h} \longrightarrow p_1 \text{ a.e in } Q_T.$$
$$s_{1,h} \longrightarrow s_1 \text{ a.e in } Q_T.$$

The identification of the limit is due to (3.13), (3.12).
The continuity of the capillary pressure function ensures that,

$$p_{2,h} \longrightarrow p_2 \text{ a.e in } Q_T,$$

the saturation equation ensures also,

$$s_{2,h} \longrightarrow s_2 \text{ a.e in } Q_T,$$

and this achieve (3.31), (3.33).
The maximum principle (3.32) and the identification (3.34) are conserved through a limit process. Finally the weak convergence (3.35) is a consequence of (3.16), and the identification of the limit is due to (3.34). $\square$

The technique for obtaining solutions of the system (1.22)–(1.23) is to pass to the limit as h goes to zero on the solutions of

$$\phi\partial_t(\tilde{r}_{i,h}) - \operatorname{div}(\mathbf{K}M_i(s_{i,h})\rho_i(p_{i,h})\nabla p_{i,h}) + \operatorname{div}(\mathbf{K}M_i(s_{i,h})\rho_i^2(p_{i,h})\mathbf{g}) \\ + (-1)^i\eta\operatorname{div}(\rho_i(p_{i,h})\nabla(p_{1,h}-p_{2,h})) + \rho_i(p_{i,h})s_{i,h}f_{P,h} = \rho_i(p_{i,h})s_i^I f_{I,h} \quad (3.40)$$

Remark that this system ($i=1,2$) is nothing else than (3.2)-(3.3), written for $n=0$ to $N-1$ by using the definition (3.10) and (3.11). Let us consider the weak formulations ($i=1,2$) on which we have to pass to the limit

$$\langle\phi\partial_t\tilde{r}_{i,h},\varphi_i\rangle + \int_{Q_T}\mathbf{K}M_i(s_{i,h})\rho_i(p_{i,h})\nabla p_{i,h}\cdot\nabla\varphi_i\,dxdt - \\ \int_{Q_T}\mathbf{K}\rho_i^2(p_{i,h})M_i(s_{i,h})\mathbf{g}\cdot\nabla\varphi_i\,dxdt - (-1)^i\eta\int_{Q_T}\rho_i(p_{i,h})\nabla(p_{1,h}-p_{2,h})\cdot\nabla\varphi_i\,dxdt \\ + \int_{Q_T}\rho_i(p_{i,h})s_{i,h}f_{P,h}\varphi_i\,dxdt = \int_{Q_T}\rho_i(p_h)s_{i,h}^I f_{I,h}\varphi_i\,dxdt. \quad (3.41)$$

where φ_i ($i=1,2$) belongs to $L^2(0,T;H^1_{\Gamma_1}(\Omega))$.
Next, we pass to the limit on each term of (3.41) which is conserved by the previous proposition.
The passage to the limit on the first term is due to (3.35), for the second term we have $M_i(s_{i,h})\rho_i(p_{i,h})\nabla\varphi_i$ converges almost everywhere in Q_T and dominated which leads by Lebesgue theorem to a strong convergence in $L^2(Q_T)$ and by virtue of the weak convergence (3.29) we establish the convergence of the second term of (3.41) to the desired term. The last three terms converge obviously to the wanted limit due to the previous proposition and Lebesgue theorem.

We then have established the weak formulation (1.25)-(1.26) of theorem 3.2. Furthermore, we have well obtained by proposition 3.3

$$0 \le s_i(t,x) \le 1 \text{ a.e. in } Q_T, \quad s_2 \in L^2(0,T;H^1_{\Gamma_1}(\Omega)), \\ p_i \in L^2(0,T;H^1_{\Gamma_1}(\Omega)), \phi\partial_t(\rho_i(p_i)s_i) \in L^2(0,T;(H^1_{\Gamma_1}(\Omega))'),\ i=1,2.$$

The compactness property on $\rho_i(p_{i,h})s_{i,h}$ implies $\rho_i(p_i)s_i \in C^0([0,T];L^2(\Omega))$, for $i=1,2$. Theorem 3.2 is then proved.

4 Proof of Theorem 3.1 (Degenerate case)

The proof is based on the existence result established for the non-degenerate case and the compactness lemma 3.8.

Lemma 3.7. *The sequences* $(s_i^\eta)_\eta$, $(p^\eta := p_2^\eta + \tilde{p}(s_1^\eta))_\eta$ *defined by the Theorem 3.2 satisfy*

$$0 \le s_i^\eta(t,x) \le 1 \quad a.e. \quad in \quad (t,x) \in Q_T \tag{4.1}$$

$$(p^\eta)_\eta \text{ is uniformly bounded in } L^2(0,T; H^1_{\Gamma_1}(\Omega)) \tag{4.2}$$

$$(\sqrt{\eta}\, \nabla f(s_1^\eta))_\eta \text{is uniformly bounded in } L^2(Q_T) \tag{4.3}$$

$$(\sqrt{M_i(s_i^\eta)}\, \nabla p_i^\eta)_\eta \text{is uniformly bounded in } L^2(Q_T) \tag{4.4}$$

$$(\beta(s_1^\eta))_\eta \text{ is uniformly bounded in } L^2(0,T; H^1(\Omega)) \tag{4.5}$$

$$(\phi \partial_t(\rho_i(p_i^\eta) s_i^\eta))_\eta \text{ is uniformly bounded in } L^2(0,T; H^1_{\Gamma_1}(\Omega)') \tag{4.6}$$

Démonstration. The maximum principle (4.1) is conserved through the limit process.
For the next three estimates, consider the $L^2(\Omega)$ scalar product of (1.22) by $g_1(p_1^\eta) = \int_0^{p_1^\eta} \frac{1}{\rho_1(\xi)}\, d\xi$ and (1.23) by $g_2(p_2^\eta) = \int_0^{p_2^\eta} \frac{1}{\rho_2(\xi)}\, d\xi$ and adding them after denoting by $\mathcal{H}_i(p_i^\eta) = \rho_i(p_i^\eta) g_i(p_i^\eta) - p_i^\eta$ $(i = 1, 2.)$, then we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi \left(s_1^\eta \mathcal{H}_1(p_1^\eta) + s_2^\eta \mathcal{H}_2(p_2^\eta) + \int_0^{s_1^\eta} f(\xi)\, d\xi \right) dx + \int_\Omega \mathbf{K} M_1(s_1^\eta) \nabla p_1^\eta \cdot \nabla p_1^\eta\, dx \\
&+ \eta \int_\Omega |\nabla f(s_1^\eta)|^2\, dx + \int_\Omega \mathbf{K} M_2(s_2^\eta) \nabla p_2^\eta \cdot \nabla p_2^\eta\, dx = \int_\Omega \mathbf{K} M_1(s_1^\eta) \rho_1(p_1^\eta) \mathbf{g} \cdot \nabla p_1^\eta\, dx \\
&+ \int_\Omega \mathbf{K} M_2(s_2^\eta) \rho_2(p_2^\eta) \mathbf{g} \cdot \nabla p_2^\eta\, dx + \int_\Omega \rho_1(p_1^\eta) s_1^I f_I g_1(p_1^\eta)\, dx - \int_\Omega \rho_1(p_1^\eta) s_1^\eta f_p g_1(p_1^\eta)\, dx \\
&- \int_\Omega \rho_2(p_2^\eta) s_2^\eta f_p g_2(p_2^\eta)\, dx + \int_\Omega \rho_2(p_2^\eta) s_2^I f_I g_2(p_2^\eta)\, dx.
\end{aligned} \tag{4.7}$$

Integrate (4.7) over $(0,T)$

$$\begin{aligned}
&\int_\Omega \phi \Big(s_1^\eta \mathcal{H}_1(p_1^\eta) + s_2^\eta \mathcal{H}_2(p_2^\eta) \Big)(x,T)\, dx + \int_{Q_T} \mathbf{K} M_1(s_1^\eta) \nabla p_1^\eta \cdot \nabla p_1^\eta\, dxdt \\
&+ \eta \int_{Q_T} |\nabla f(s_1^\eta)|^2\, dxdt + \int_{Q_T} \mathbf{K} M_2(s_2^\eta) \nabla p_2^\eta \cdot \nabla p_2^\eta\, dxdt \\
&= \int_\Omega \phi \Big(s_1^0 \mathcal{H}_1(p_1^0) + s_2^0 \mathcal{H}_2(p_2^0) \Big)\, dx - \int_\Omega \int_{s_1^0}^{s_1^\eta(x,T)} f(\xi)\, d\xi dx \\
&+ \int_{Q_T} \mathbf{K} M_1(s_1^\eta) \rho_1(p_1^\eta) \mathbf{g} \cdot \nabla p_1^\eta\, dxdt + \int_{Q_T} \mathbf{K} M_2(s_2^\eta) \rho_2(p_2^\eta) \mathbf{g} \cdot \nabla p_2^\eta\, dxdt \\
&+ \int_{Q_T} \rho_1(p_1^\eta) s_1^I f_I g_1(p_1^\eta)\, dxdt - \int_{Q_T} \rho_1(p_1^\eta) s_1^\eta f_p g_1(p_1^\eta)\, dxdt \\
&- \int_{Q_T} \rho_2(p_2^\eta) s_2^\eta f_p g_2(p_2^\eta)\, dxdt + \int_{Q_T} \rho_2(p_2^\eta) s_2^I f_I g_2(p_2^\eta)\, dxdt.
\end{aligned} \tag{4.8}$$

The first term on the left hand side of (4.8) is positive and the two first terms on the right hand side are bounded since $p_i^0 \in L^2(\Omega)$ and $0 \le s_i^0 \le 1$. The third and the fourth terms on the right hand side, corresponding to gravity term, can be absorbed by the degenerate dissipative term on pressures (namely the second and fourth terms on the left hand side of (4.8)) since :

$$\left|\int_{Q_T} \mathbf{K} M_i(s_i^\eta)\rho_i(p_i^\eta)\mathbf{g}\cdot\nabla p_i^\eta\,dxdt\right| \le C + \frac{k_0}{2}\int_{Q_T} M_i(s_i^\eta)|\nabla p_i^\eta|^2\,dxdt, \quad i=1,2.$$

Finally, using the fact that the functions g_i ($i=1,2.$) are sublinear, we deduce from (4.8) that

$$\int_{Q_T} M_1(s_1^\eta)|\nabla p_1^\eta|^2\,dxdt + \int_{Q_T} M_2(s_2^\eta)|\nabla p_2^\eta|^2\,dxdt + \eta\int_{Q_T}\nabla f(s_1^\eta)\cdot\nabla f(s_1^\eta)\,dxdt$$
$$\le C(1+\parallel p_1^\eta\parallel_{L^2(Q_T)} + \parallel p_2^\eta\parallel_{L^2(Q_T)}), \quad (4.9)$$

where C is a constant independent of η.
From the definition of the global pressure, we have

$$\nabla p^\eta = \nabla p_2^\eta + \frac{M_1(s_1^\eta)}{M(s_1^\eta)}\nabla f(s_1^\eta) = \nabla p_1^\eta - \frac{M_2(s_2^\eta)}{M(s_1^\eta)}\nabla f(s_1^\eta), \quad (4.10)$$

and consequently

$$\int_{Q_T} M(s_1^\eta)|\nabla p^\eta|^2\,dxdt + \int_{Q_T}\frac{M_1(s_1^\eta)M_2(s_2^\eta)}{M(s_1^\eta)}|\nabla f(s_1^\eta)|^2\,dxdt$$
$$= \int_{Q_T} M_1(s_1^\eta)\nabla p_1^\eta\cdot\nabla p_1^\eta\,dxdt + \int_{Q_T} M_2(s_2^\eta)\nabla p_2^\eta\cdot\nabla p_2^\eta\,dxdt. \quad (4.11)$$

On the other hand

$$\parallel p_1^\eta\parallel_{L^2(Q_T)}\le\parallel p^\eta\parallel_{L^2(Q_T)} + \parallel \bar{p}(s_1^\eta)\parallel_{L^2(Q_T)}\le C\parallel\nabla p^\eta\parallel_{L^2(Q_T)} + \parallel\bar{p}(s_1^\eta)\parallel_{L^2(Q_T)},$$

due to Poincaré's inequality, in the same way we have

$$\parallel p_2^\eta\parallel_{L^2(Q_T)}\le C\parallel\nabla p^\eta\parallel_{L^2(Q_T)} + \parallel\tilde{p}(s_1^\eta)\parallel_{L^2(Q_T)}.$$

From the above estimates and (4.11), the estimate (4.13) yields

$$\int_{Q_T} M(s_1^\eta)|\nabla p^\eta|^2\,dxdt + \int_{Q_T}\frac{M_1(s_1^\eta)M_2(s_2^\eta)}{M(s_1^\eta)}|\nabla f(s_1^\eta)|^2\,dxdt$$
$$+\eta\int_{Q_T}\nabla f(s_1^\eta)\cdot\nabla f(s_1^\eta)\,dxdt \le C(1+\parallel\nabla p^\eta\parallel_{L^2(Q_T)}). \quad (4.12)$$

The Young inequality permits to absorb the last term by the first term on the left hand side of (4.12) to get

$$\int_{Q_T} M(s_1^\eta)|\nabla p^\eta|^2\,dxdt + \int_{Q_T} \frac{M_1(s_1^\eta)M_2(s_2^\eta)}{M(s_1^\eta)}|\nabla f(s_1^\eta)|^2\,dxdt \\ + \int_{Q_T} M_1(s_1^\eta)\nabla p_1^\eta\cdot\nabla p_1^\eta\,dxdt + \int_{Q_T} M_2(s_2^\eta)\nabla p_2^\eta\cdot\nabla p_2^\eta\,dxdt \\ + \eta\int_{Q_T}\nabla f(s_1^\eta)\cdot\nabla f(s_1^\eta)\,dxdt \le C, \quad (4.13)$$

where C is a constant independent of η. Thus, the estimates (4.3)–(4.4) are established. The estimate (4.5) is also a consequence of (4.13) since

$$\int_{Q_T}|\nabla\beta(s_1^\eta)|^2\,dxdt = \int_{Q_T}\frac{M_1^2(s_1^\eta)M_2^2(s_2^\eta)}{M^2(s_1^\eta)}|\nabla f(s_1^\eta)|^2\,dxdt \quad (4.14)$$

$$\le \int_{Q_T} M_1(s_1^\eta)M_2(s_2^\eta)|\nabla f(s_1^\eta)|^2\,dxdt \le C. \quad (4.15)$$

For the last estimate (4.6), let $\varphi_i \in L^2(0,T;H^1_{\Gamma_1}(\Omega))$ and denote the bracket $\langle\cdot,\cdot\rangle$ the duality product between $L^2(0,T;(H^1_{\Gamma_1}(\Omega))')$ and $L^2(0,T;H^1_{\Gamma_1}(\Omega))$, using (4.10), one gets

$$|\langle\phi\partial_t(\rho_i(p_i^\eta)s_i^\eta),\varphi_i\rangle| \le \left|\eta\int_{Q_T}\rho_i(p_i^\eta)\nabla f(s_i^\eta)\cdot\nabla\varphi_i\,dxdt\right| \\ + \left|\int_{Q_T}\mathbf{K}\rho_i(p_i^\eta)(M_i(s_i^\eta)\nabla p^\eta + \nabla\beta(s_1^\eta))\cdot\nabla\varphi_i\,dxdt\right| \\ + \left|\int_{Q_T}\mathbf{K}\rho_i^2(p_i^\eta)M_i(s_i^\eta)\mathbf{g}\cdot\nabla\varphi_i\,dxdt\right| + \left|\int_{Q_T}\rho_i(p_i^\eta)s_i^\eta f_P\varphi_i\,dxdt\right| \\ + \left|\int_{Q_T}\rho_i(p_i^\eta)s_i^I f_I\varphi_i\,dxdt\right|,$$

and from the estimates (4.2)–(4.5), we deduce

$$|\langle\phi\partial_t(\rho_i(p_i^\eta)s_i^\eta),\varphi_i\rangle| \le C \parallel\varphi_i\parallel_{L^2(0,T;H^1_{\Gamma_1}(\Omega))},$$

which establish (4.6) and complete the proof of the lemma. □

Lemma 3.8. *(Compactness result for degenerate case)*
For every M, the following implicit set

$$E_M = \{(\rho_1(p_1)s_1,\rho_2(p_2)s_2)\in L^2(Q_T)\times L^2(Q_T),\ \textit{such that} \\ \|\beta(s_1)\|_{L^2(0,T;H^1(\Omega))}\le M, \\ \|\sqrt{M_1(s_1)}\nabla p_1\|_{L^2(Q_T)} + \|\sqrt{M_2(s_2)}\nabla p_2\|_{L^2(Q_T)} \le M, \\ \|\phi\partial_t(\rho_1(p_1)s_1)\|_{L^2(0,T;(H^1_{\Gamma_1}(\Omega))')} + \|\phi\partial_t(\rho_2(p_2)s_2)\|_{L^2(0,T;(H^1_{\Gamma_1}(\Omega))')} \le M\}$$

is relatively compact in $L^2(Q_T) \times L^2(Q_T)$, and $\gamma(E_M)$ is relatively compact in $L^2(\Sigma_T) \times L^2(\Sigma_T)$, ($\gamma$ denotes the trace on Σ_T operator).

Démonstration. The proof is inspired by the compactness lemma 4.3 ([45], p. 37) which introduced for compressible degenerate model. We generalize this result for our compressible degenerate model. Denote by

$$u = \rho_1(p_1)s_1, \;\; v = \rho_2(p_2)(1 - s_1).$$

Define the map $\mathbb{H} : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+ \times [0, \beta(1)]$ defined by

$$\mathbb{H}(u, v) = (p, \beta(s_1)) \tag{4.16}$$

where u and v are solutions of the system

$$u(p, \beta(s_1)) = \rho_1(p - \bar{p}(\beta^{-1}(\beta(s_1))))\beta^{-1}(\beta(s_1))$$

$$v(p, \beta(s_1)) = \rho_2(p - \tilde{p}(\beta^{-1}(\beta(s_1))))(1 - \beta^{-1}(\beta(s_1)).$$

Note that $\mathbb{H}$ is well defined as a diffeomorphism, since

$$\begin{aligned}
\frac{\partial u}{\partial p} &= \rho_1'(p - \bar{p}(\beta^{-1}(\beta(s_1))))\beta^{-1}(\beta(s_1)) \geq 0 \\
\frac{\partial u}{\partial \beta} &= \rho_1'(p - \bar{p}(\beta^{-1}(\beta(s_1))))[-\bar{p}'(\beta^{-1}(\beta(s_1)))(\beta^{-1\prime}(\beta(s_1)))]\beta^{-1}(\beta(s_1)) \\
&+ \rho_1(p - \bar{p}(\beta^{-1}(\beta(s_1))))\beta^{-1\prime}(\beta(s_1)) \geq 0 \\
\frac{\partial v}{\partial p} &= -\rho_2'(p - \tilde{p}(\beta^{-1}(\beta(s_1))))(1 - \beta^{-1}(\beta(s_1))) \geq 0 \\
\frac{\partial v}{\partial \beta} &= \rho_2'(p - \tilde{p}(\beta^{-1}(\beta(s_1))))[-\tilde{p}'(\beta^{-1}(\beta(s_1)))(\beta^{-1\prime}(\beta(s_1)))][1 - \beta^{-1}(\beta(s_1))] \\
&- \rho_2(p - \tilde{p}(\beta^{-1}(\beta(s_1))))\beta^{-1\prime}(\beta(s_1)) \leq 0,
\end{aligned}$$

and if one of the saturations is zero the other one is one, this conserves that the jacobian determinant of the map $\mathbb{H}^{-1}$ is strictly negative.
Furthermore, $\mathbb{H}^{-1}$ is an Hölder function, in sense that u and v are Hölder functions of order θ with $0 < \theta \leq 1$. For that, let (q_1, σ_1) and (q_2, σ_2) in $\mathbb{R}^+ \times [0, \beta(1)]$, we

have

$$\begin{aligned}
&|u(q_1,\sigma_1)-u(q_2,\sigma_2)|\\
&=|\rho_1(q_1-\bar p(\beta^{-1}(\sigma_1)))\beta^{-1}(\sigma_1)-\rho_1(q_2-\bar p(\beta^{-1}(\sigma_2)))\beta^{-1}(\sigma_2)|\\
&\le|\rho_1(q_1-\bar p(\beta^{-1}(\sigma_1)))-\rho_1(q_2-\bar p(\beta^{-1}(\sigma_2)))|+\rho_M|\beta^{-1}(\sigma_1)-\beta^{-1}(\sigma_2)|,
\end{aligned}$$

since β^{-1} is an Hölder function of order θ, $0<\theta\le 1$, and the map ρ_1 is bounded and of class C^1, we deduce up to two cases :
The first case $|q_1-q_2|\ge 1$,

$$\begin{aligned}
&|u(q_1,\sigma_1)-u(q_2,\sigma_2)|\\
&\le|\rho_1(q_1-\bar p(\beta^{-1}(\sigma_1)))-\rho_1(q_2-\bar p(\beta^{-1}(\sigma_2)))|+\rho_M|\beta^{-1}(\sigma_1)-\beta^{-1}(\sigma_2)|\\
&\le\rho_M+\rho_M|\beta^{-1}(\sigma_1)-\beta^{-1}(\sigma_2)|\\
&\le\rho_M|q_1-q_2|^\theta+\rho_M C_\beta|\sigma_1-\sigma_2|^\theta,
\end{aligned}$$

for the other case $|q_1-q_2|<1$ we have,

$$\begin{aligned}
&|u(q_1,\sigma_1)-u(q_2,\sigma_2)|\\
&\le|\rho_1(q_1-\bar p(\beta^{-1}(\sigma_1)))-\rho_1(q_2-\bar p(\beta^{-1}(\sigma_2)))|+\rho_M|\beta^{-1}(\sigma_1)-\beta^{-1}(\sigma_2)|\\
&\le C(|q_1-q_2|+|\bar p(\beta^{-1}(\sigma_1))-\bar p(\beta^{-1}(\sigma_2))|)+\rho_M C_\beta|\sigma_1-\sigma_2|^\theta\\
&\le C|q_1-q_2|^\theta+C|\bar p(\beta^{-1}(\sigma_1))-\bar p(\beta^{-1}(\sigma_2))|+\rho_M C_\beta|\sigma_1-\sigma_2|^\theta
\end{aligned}$$

further more one can easily show that $\bar p$ is a $C^1([0,1];\mathbb{R})$, it follows that

$$|u(q_1,\sigma_1)-u(q_2,\sigma_2)|\le C|q_1-q_2|^\theta+C|\sigma_1-\sigma_2|^\theta. \tag{4.17}$$

In the same way, we have

$$|v(q_1,\sigma_1)-v(q_2,\sigma_2)|\le c_1|q_1-q_2|^\theta+c_2|\sigma_1-\sigma_2|^\theta. \tag{4.18}$$

For $0<\tau<1$, and $1<r<\infty$, let us denote

$$W^{\tau,r}(\Omega)=\{w\in L^r(\Omega);\int_\Omega\int_\Omega\frac{|w(x)-w(y)|^r}{|x-y|^{d+\tau r}}\,dxdy<+\infty\}$$

equipped with the norm

$$\|w\|_{W^{\tau,r}(\Omega)}=\Big(\|w\|^r_{L^r(\Omega)}+\int_\Omega\int_\Omega\frac{|w(x)-w(y)|^r}{|x-y|^{d+\tau r}}\,dxdy\Big)^{\frac{1}{r}},$$

recall d denote the space dimension. Let q, σ be in $W^{\tau,r}(\Omega)\times W^{\tau,r}(\Omega)$, then the

Hölder functions u and v belong to $W^{\theta\tau,r/\theta}(\Omega)$. In fact, we have

$$|u(q,\sigma)| \leq c_1|q|^\theta + c_2|\sigma|^\theta,$$

then u belongs to $L^{r/\theta}(\Omega)$. Furthermore,

$$\int_\Omega\int_\Omega \frac{|u(q(x),\sigma(x)) - u(q(y),\sigma(y))|^{r/\theta}}{|x-y|^{d+\tau r}}\,dxdy \leq c_1^{r/\theta}\int_\Omega\int_\Omega \frac{|q(x)-q(y)|^r}{|x-y|^{d+\tau r}}\,dxdy$$
$$+c_2^{r/\theta}\int_\Omega\int_\Omega \frac{|\sigma(x)-\sigma(y)|^r}{|x-y|^{d+\tau r}}\,dxdy,$$

which ensures,

$$\|u(q,\sigma)\|_{W^{\theta\tau,r/\theta}(\Omega)} \leq c(\|q\|^\theta_{W^{\tau,r}(\Omega)} + \|\sigma\|^\theta_{W^{\tau,r}(\Omega)}).$$

Using the continuity of the injection of $H^1(\Omega)$ into $W^{\tau,2}(\Omega)$, with $\tau < 1$,

$$\|u(p,\beta(s_1))\|_{W^{\theta\tau,2/\theta}(\Omega)} \leq c(\|p\|^\theta_{W^{\tau,2}(\Omega)} + \|\beta(s_1)\|^\theta_{W^{\tau,r}(\Omega)}) \leq c(\|p\|^\theta_{H^1(\Omega)} + \|\beta(s_1)\|^\theta_{H^1(\Omega)})$$

integrating the above inequality over $(0,T)$,

$$\|u(p,\beta(s_1))\|_{L^{2/\theta}(0,T;W^{\theta\tau,2/\theta}(\Omega))} \leq c\|p\|^\theta_{L^2(0,T;H^1(\Omega))} + \|\beta(s_1)\|^\theta_{L^2(0,T;H^1(\Omega))}$$

Furthermore the porosity function ϕ belongs to $W^{1,\infty}(\Omega)$, it follows that

$$\|\phi u(p,\beta(s_1))\|_{L^{2/\theta}(0,T;W^{\theta\tau,2/\theta}(\Omega))} \leq C.$$

As Ω is bounded and regular, we have, for $\tau' < \theta\tau$,

$$W^{\theta\tau,2/\theta}(\Omega) \subset W^{\tau',2/\theta}(\Omega) \subset (H^1_{\Gamma_1}(\Omega))'$$

with compact injection from $W^{\theta\tau,2/\theta}(\Omega)$ into $W^{\tau',2/\theta}(\Omega)$.
Finally, from a standard compactness argument, we get

$$E_M \text{ is relatively compact in } L^{2/\theta}(0,T;W^{\tau',2/\theta}(\Omega)) \subset L^2(0,T;L^2(\Omega)).$$

Secondly, the trace operator γ maps continuously $W^{\tau',2/\theta}(\Omega))$ into $W^{\tau'-\theta/2,2/\theta}(\Gamma))$ as soon as $\tau' > \theta/2$. Choosing for example $\tau' = \frac{3\theta}{4}$, we deduce the relative compactness of $\gamma(E_M)$ into $L^2(\Sigma_T) \times L^2(\Sigma_T)$.
This closes the proof of lemma 3.8. □

From the previous two lemmas, we deduce the following convergences.
Lemma 3.9. *(Strong and weak convergences)*

Up to a subsequence the sequence $(s_i^\eta)_\eta$, $(p^\eta)_\eta$, $(p_i^\eta)_\eta$ verify the following convergence

$$
\begin{aligned}
&p^\eta \longrightarrow p && \textit{weakly in } L^2(0,T;H^1_{\Gamma_1}(\Omega)), && (4.19)\\
&\beta(s_1^\eta) \longrightarrow \beta(s_1) && \textit{weakly in } L^2(0,T;H^1(\Omega)), && (4.20)\\
&p^\eta \longrightarrow p && \textit{almost everywhere in } Q_T && (4.21)\\
&s_1^\eta \longrightarrow s_1 && \textit{almost everywhere in } Q_T \textit{ and } \Sigma_T && (4.22)\\
&s_1^\eta \longrightarrow s_1 && \textit{strongly in } L^2(Q_T) \textit{ and } L^2(\Sigma_T) && (4.23)\\
&0 \le s_i(t,x) \le 1 && \textit{almost everywhere in } (t,x)\in Q_T, && (4.24)\\
&p_i^\eta \longrightarrow p_i && \textit{almost everywhere in } Q_T && (4.25)\\
&\phi\partial_t(\rho_i(p_i^\eta)s_i^\eta) \longrightarrow \phi\partial_t(\rho_i(p_i)s_i) && \textit{weakly in } L^2(0,T;(H^1_{\Gamma_1}(\Omega))'). && (4.26)
\end{aligned}
$$

Démonstration. The weak convergences (4.19)–(4.20) follows from the uniform estimates (4.2) and (4.5) of lemma 3.7.
The lemma 3.8 ensures the following strong convergences

$$
\begin{aligned}
\rho_i(p_i^\eta)s_i^\eta &\longrightarrow l_i \text{ in } L^2(Q_T) \text{ and a. e. in } Q_T,\\
\rho_i(p_i^\eta)s_i^\eta &\longrightarrow l_i \text{ in } L^2(\Sigma_T) \text{ and a. e. in } \Sigma_T,
\end{aligned}
$$

As the map $\mathbb{H}$ defined in (4.16) is continuous, we deduce

$$
\begin{aligned}
p^\eta &\longrightarrow p \text{ a. e. in } Q_T \text{ and a. e. in } \Sigma_T,\\
\beta(s_1^\eta) &\longrightarrow \beta^\star \text{ a. e. in } Q_T \text{ and a. e. in } \Sigma_T.
\end{aligned}
$$

The convergence (4.21) is then established and as β^{-1} is continuous,

$$
s_1^\eta \longrightarrow s_1 = \beta^{-1}(\beta^\star) \text{ a. e. in } Q_T \text{ and a. e. in } \Sigma_T.
$$

From (4.1), the estimate (4.24) holds and the Lebesgue theorem ensures the strong convergence (4.23).
The convergence (4.25) is a consequence of (4.21)–(4.22).
At last, the weak convergence (4.26) is a consequence of the estimate (4.6), and the identification of the limit follows from the previous convergence. □

In order to achieve the proof of Theorem 3.1, it remains to pass to the limit as η goes to zero in the formulations (1.25)–(1.26), for all smooth test functions

$\varphi \in C^1([0,T]; H^1_{\Gamma_1}(\Omega)) \cap L^2(0,T; H^2(\Omega))$ such that $\varphi(T) = 0$

$$
\begin{aligned}
-\int_{Q_T} \phi \rho_i(p_i^\eta) s_i^\eta \partial_t \varphi \, dxdt + \int_{Q_T} \mathbf{K} M_i(s_i^\eta) \rho_i(p_i^\eta) \nabla p_i^\eta \cdot \nabla \varphi \, dxdt \\
- \int_{Q_T} \mathbf{K} M_i(s_i^\eta) \rho_i^2(p_i^\eta) \mathbf{g} \cdot \nabla \varphi \, dxdt + \int_{Q_T} \rho_i(p_i^\eta) s_i^\eta f_P \varphi \, dxdt \\
- (-1)^i \, \eta \int_\Omega \rho_i(p_i^\eta) \nabla (p_1^\eta - p_2^\eta) \cdot \nabla \varphi \, dxdt = \int_{Q_T} \rho_i(p_i^\eta) s_i^I f_I \varphi \, dxdt \\
+ \int_\Omega \phi \rho_i(p_i^0) s_i^0 \varphi(0,x) \, dxdt, \quad i = 1,2. \quad (4.27)
\end{aligned}
$$

The first term converges due to the strong convergence of $\rho_i(p_i^\eta) s_i^\eta$ to $\rho_i(p_i) s_i$ in $L^2(Q_T)$.
The second term can be written, with the help of global pressure, as,

$$
\begin{aligned}
\int_{Q_T} \mathbf{K} M_i(s_i^\eta) \rho_i(p_i^\eta) \nabla p_i^\eta \cdot \nabla \varphi \, dxdt = \int_{Q_T} \mathbf{K} M_i(s_i^\eta) \rho_i(p_i^\eta) \nabla p^\eta \cdot \nabla \varphi \, dxdt \\
+ \int_{Q_T} \mathbf{K} \rho_i(p_i^\eta) \nabla \beta(s_i^\eta) \cdot \nabla \varphi \, dxdt. \quad (4.28)
\end{aligned}
$$

The two terms on the right hand side of the equation (4.28) converge arguing in two steps. Firstly, the Lebsgue theorem and the convergences (4.22)(4.25) establish

$$
\begin{aligned}
\rho_i(p_i^\eta) M_i(s_i^\eta) \nabla \varphi \longrightarrow \rho_i(p_i) M_i(s_i) \nabla \varphi \text{ strongly in } (L^2(Q_T))^d, \\
\rho_i(p_i^\eta) \nabla \varphi \longrightarrow \rho_i(p_i) \nabla \varphi \text{ strongly in } (L^2(Q_T))^d.
\end{aligned}
$$

Secondly, the weak convergence on global pressure (4.19) and the weak convergence (4.20) combined to the above strong convergences allow the convergence for the terms of the right hand side of (4.28).
The fifth term can be written as,

$$
\eta \int_\Omega \rho_i(p_i^\eta) \nabla (p_1^\eta - p_2^\eta) \cdot \nabla \varphi \, dxdt = \sqrt{\eta} \int_\Omega \rho_i(p_i^\eta) (\sqrt{\eta} \nabla f(s_1^\eta)) \cdot \nabla \varphi \, dxdt, \quad (4.29)
$$

The Cauchy-Schwarz inequality and the uniform estimate (4.3) ensure the convergence of this term to zero as η goes to zero.
The other terms converge using (4.22)(4.25) and the Lebesgue dominated convergence theorem.
The weak formulations (1.12) and (1.13) are then established.
The main theorem 3.1 is then established.

CHAPITRE 4

CONVERGENCE OF A FINITE VOLUME SCHEME FOR GAS WATER FLOW IN A MULTI-DIMENSIONAL POROUS MEDIA

Abstract. A classical model for water-gas flows in porous media is considered. The degenerate coupled system of equations obtained by mass conservation is usually approximated by finite volume schemes in the oil reservoir simulations. The convergence properties of these schemes are only known for incompressible fluids. This chapter deals with construction and convergence analysis of a finite volume scheme for compressible and immiscible flow in porous media. In comparison with incompressible fluid, compressible fluids requires more powerful techniques. We present a new result of convergence in a two or three dimensional porous medium and under the only modification that the density of gas depends on global pressure.

1 Introduction

Mathematical study of a petroleum-engineering schemes takes an important place in oil recovery engineering for production of hydrocarbons from petroleum reservoirs. In soil mechanics, engineers study the air-water flow in soils and they prefer the use of a two phase flow model. More recently, due to the effects of global warming on climate change, two and multi-phase flow has been receiving an increasing attention in connection with the disposal of radioactive waste and sequestration of CO_2.

It has been shown that the governing equations describing two incompressible

(compressible) phase flow in porous media can be written in a fractional flow formulation, i.e., in terms of global pressure and saturation and that formulation has been studied ; For incompressible flow and from a mathematical point of view [2, 22] and it has been used in numerical codes [23, 24, 21]. For immiscible and compressible two-phase flows (e.g., air, water), Ewing and al. in ([38], [27]) follow the ideas of Chavent by considering global pressure and saturation as unknowns of the system. This formulation leads to a global pressure equation coupled to the water saturation equation. The authors proposed a finite element and finite difference method to solve the saturation equation and a mixed finite element to approximate the global pressure equation. Note that the global pressure reads as a parabolic equation with a source term involving the evolution of the capillary pressure term. This evolution term is approached by Picard iterations. This algorithm converges numerically and suggests a continuous dependence on the capillary terms and legitimates some approximations for small capillary pressure. Further, it has been proven that this fractional flow approach is far more efficient than the original two-pressure approach from the computational point of view [26] and the references cited therein. For compressible flow and from mathematical point of view, the fractional flow formulation is sufficient enough at least for slightly compressible gas, i.e, when the density of gas depends on the global pressure [45, 47]. More recently and under the context of theoretical study of compressible flow in porous media, the two-pressure approach has been treated by Z. Khalil, M. Saad [52, 54].
In this paper, we consider immiscible two-phase flows ; the gas phase is considered to be compressible and the water one to be incompressible. The model is derived by using the global pressure notation and is justified at least for slightly compressible gas. The system represents two kinds of degeneracy. The first one is the classical degeneracy of the diffusion operator in saturation due to the capillary effect. The second one represents a degeneracy in the evolution term in pressure occurring in the region where the gas saturation vanishes : A classical compactness result on pressure is missed in the region where the gas phase is missing.
The aim of the present paper is to show that the approximate solutions obtained with the proposed upwind finite volume scheme (3.5)-(3.6), converges as the mesh size tends to zero, to a solution of system (2.1)-(2.2) in an appropriate sense defined in Section 2. In Section 3 we introduce the finite volume discretization, the numerical scheme and state the main convergence results. In Section 4, maximum principle on saturation is attained and *a priori* estimates on the discrete gradient of the capillary term and on the discrete gradient of the global pressure are derived as the continuous case in C. Galusinski, M. Saad [47]. In Section 5, a well posedness of the scheme is inspired by H. W. Alt, S. Luckhaus[3]. Section 6 is devoted to a space-time L^1 compactness argument, in this section we follow B. Andreianov,

M. Bendahmane and R. Ruiz-Baier []. Finally, the passage to the limit on the scheme needs a powerful techniques due to the lack of compactness result on global pressure in the region where the saturation of gas vanishes, and this performed in section 7.

2 The mathematical formulation

The fractional flow formulation describing the immiscible displacement of two compressible and incompressible fluids are given by the following mass conservation of each phase [47] :

$$\begin{aligned}\partial_t(\phi\rho(p)s) - \operatorname{div}(\mathbf{K}\rho(p)M_1(s)\nabla p) - \operatorname{div}(\mathbf{K}\rho(p)\alpha(s)\nabla s)\\ + \operatorname{div}(\mathbf{K}\rho^2(p)M_1(s)\mathbf{g}) + \rho(p)sf_P = 0,\end{aligned} \tag{2.1}$$

$$\partial_t(\phi s) + \operatorname{div}(\mathbf{K}M_2(s)\nabla p) - \operatorname{div}(\mathbf{K}\alpha(s)\nabla s) + \operatorname{div}(\mathbf{K}\rho_2 M_2(s)\mathbf{g}) + sf_P = f_P - f_I. \tag{2.2}$$

where ϕ and $\mathbf{K}$ are the porosity and absolute permeability of the porous medium ; ρ, ρ_2, p and s are respectively the densities of gas and water (density of water is constant), the global pressure and the saturation of gas ; f_P, f_I, M_1, M_2 and $\mathbf{g}$ are respectively the production and injection source terms, the mobilities of gas and water and the gravity term.
To define the capillary term α, let us denote by p_1, p_2 to be respectively the pressures of gas and water phases. Thus, we define the capillary pressure and the total mobility as

$$p_{12}(s(t,x)) = p_1(t,x) - p_2(t,x) \tag{2.3}$$

$$M(s) = M_1(s) + M_2(s) \tag{2.4}$$

and the function $s \mapsto p_{12}(s)$ is non-decreasing ($\frac{dp_{12}}{ds}(s) \geq 0,$ for all $s \in [0,1]$). In this paper, the forced displacement of fluids is modellized. It is used in many enhanced recovery processes : a fluid, such as water, is injected into some wells in a reservoir while the resident hydrocarbons are produced from other wells. Now we define the capillary term

$$\alpha(s) = \frac{M_1(s)M_2(s)}{M(s)}\frac{dp_{12}}{ds}(s) \geq 0$$

defining a function $\tilde{p}(s)$ such that $\frac{d\tilde{p}}{ds}(s) = \frac{M_1(s)}{M(s)}\frac{dp_{12}}{ds}(s)$, and setting $p = p_2 + \tilde{p}$, named global pressure [22]. Thus, each phase velocity given by Darcy's law can be written as

$$\mathbf{V}_1 = -\mathbf{K}M_1(s)\nabla p - \mathbf{K}\alpha(s)\nabla s + \mathbf{K}M_1(s)\rho_1(p)\mathbf{g} \tag{2.5}$$

$$\mathbf{V}_2 = -\mathbf{K}M_2(s)\nabla p + \mathbf{K}\alpha(s)\nabla s + \mathbf{K}M_2(s)\rho_2\mathbf{g}. \tag{2.6}$$

Note that this system is strongly degenerate. In fact, the lack of coercivity of the degenerate diffusion term $\mathrm{div}(\mathbf{K}\alpha(s)\nabla s)$ is classical for incompressible flows. An additional difficulty is due to the degeneracy of the time derivative term $\phi(x)\partial_t(\rho(p)s)$ which vanishes in the region where $s = 0$. Another difficulty seems to be the degenerate diffusive pressure term $\mathrm{div}(\mathbf{K}\rho(p)M_1(s)\nabla p)$ in (2.1) where the mobility of the gas phase M_1 vanishes in $s = 0$. In fact, a pressure diffusion term appears also in the saturation equation (2.2) with the term $\mathrm{div}(\mathbf{K}M_2(s)\nabla p)$. An energy estimate coupling the two equations (2.1)-(2.2) lets appear a non-degenerate dissipative pressure term (see section 4).
Consider a fixed time $T > 0$ and let Ω be a bounded set of $\mathbb{R}^d$ ($d \geq 1$). We set $Q_T = (0,T)\times\Omega$, $\Sigma_T = (0,T)\times\partial\Omega$. To the system (2.1)-(2.2) we add the following mixed boundary conditions and initial conditions. We consider the boundary $\partial\Omega = \Gamma_w \cup \Gamma_i$, where Γ_w denotes the water injection boundary and Γ_i the impervious one.

$$\begin{cases} s(t,x) = 0, \ \ p(t,x) = 0 \text{ on } \Gamma_w \\ \mathbf{V}_1 \cdot \mathbf{n} = \mathbf{V}_2 \cdot \mathbf{n} = 0 \text{ on } \Gamma_i, \end{cases} \tag{2.7}$$

where $\mathbf{n}$ is the outward normal to the boundary Γ_i. We force a constant pressure (shifted at zero) along the time on the region of water injection.
Initial condition :

$$\begin{cases} s(0,x) = s_0(x), \ \text{in } \Omega \\ p(0,x) = p_0(x) \text{ in } \Omega \end{cases} \tag{2.8}$$

We are going to construct a finite volume scheme on orthogonal admissible mesh, we treat here the case where

$$K = k\mathcal{I}_d$$

where k is a constant positive. For clarity, we take $k = 1$ which equivalent to change the scale in time. In remark 4.3 we give the scheme where k is a function depending on space.
Next we introduce some physically relevant assumptions on the coefficients of the system.

(H1) The porosity ϕ belongs to $L^\infty(\Omega)$ and there exist two positive constants ϕ_0 and ϕ_1 such that $\phi_0 \leq \phi(x) \leq \phi_1$ a.e. $x \in \Omega$.

(H2) The functions M_1 and M_2 belong to $\mathcal{C}^0([0,1];\mathbb{R}^+)$, $M_1(0) = 0$ and $M_2(1) = 0$.

In addition, there is a positive constant m_0 such that, for all $s \in [0,1]$,

$$M_1(s) + M_2(s) \geq m_0.$$

(H3) The function $\alpha \in \mathcal{C}^2([0,1];\mathbb{R}^+)$ satisfies $\alpha(s) > 0$ for $0 < s \leq 1$, and $\alpha(0) = 0$. We define $\beta(s) = \int_0^s \alpha(z)dz$ and assume that β^{-1} is an Hölder function of order θ, with $0 < \theta \leq 1$ on $[0, \beta(1)]$. This means that there exists a positive c such that for all $s_1, s_2 \in [0, \beta(1)]$, one has $|\beta^{-1}(s_1) - \beta^{-1}(s_1)| \leq c|s_1 - s_2|^\theta$.

(H4) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \geq 0$ a.e. $(t,x) \in Q_T$

(H5) The density ρ is a $\mathcal{C}^1(\mathbb{R})$ function, increasing, and there exist $\rho_m > 0$, $\rho_M < +\infty$ such that
$\rho_m \leq \rho(p) \leq \rho_M$ for all $p \in \mathbb{R}$.

The assumptions (H1)–(H5) are classical for porous media. Especially, a practical sufficient condition to handle (H3) is to consider that α is an Hölder function at $s = 0$. This contains several relevant physical cases of two-phase flows in porous media (see [22, chapter V]).

Define

$$H^1_{\Gamma_w}(\Omega) = \{\, u \in H^1(\Omega)\,;\, u = 0 \text{ on } \Gamma_w \,\},$$

this is an Hilbert space when equipped with the norm $\|u\|_{H^1_{\Gamma_w}(\Omega)} = \|\nabla u\|_{(L^2(\Omega))^d}$.
In the next section we introduce the existence of solutions to system (2.1)-(2.2) under the conditions (H1)–(H5).

Definition 4.1. *(Weak solutions)*
Let (H1)-(H5) hold. Assume p_0 (defined by (2.8)) belongs to $L^2(\Omega)$, and s_0 satisfies $0 \leq s_0 \leq 1$ almost everywhere in Ω. Then, the pair (s,p) is a weak solution of the problem (2.1)-(2.2) if

$$0 \leq s \leq 1 \ \textit{a.e. in } Q_T, \quad \beta(s) \in L^2(0,T; H^1_{\Gamma_w}(\Omega)), \quad p \in L^2(0,T; H^1_{\Gamma_w}(\Omega)),$$

such that for all $\varphi, \xi \in \mathcal{D}\Big([0,T) \times \Omega\Big)$,

$$\begin{aligned} &- \int_{Q_T} \phi\rho(p) s \partial_t \varphi \, dxdt - \int_\Omega \phi(x) u_0(x) \varphi(0,x)\, dx \\ &\qquad + \int_{Q_T} \rho(p) M_1(s) \nabla p \cdot \nabla \varphi \, dxdt + \int_{Q_T} \rho(p) \nabla \beta(s) \cdot \nabla \varphi \, dxdt \\ &\qquad\qquad - \int_{Q_T} \rho^2(p) M_1(s) \mathbf{g} \cdot \nabla \varphi \, dxdt + \int_{Q_T} \rho(p) s f_P \varphi \, dxdt = 0, \quad (2.9) \end{aligned}$$

$$
\begin{aligned}
-\int_{Q_T} \phi s \partial_t \xi \, dxdt - \int_\Omega \phi s_0(x) \xi(0,x)\, dx + \int_{Q_T} \nabla \beta(s) \cdot \nabla \xi \, dxdt \\
- \int_{Q_T} M_2(s) \nabla p \cdot \nabla \xi \, dxdt - \int_{Q_T} \rho_2 M_2(s) \mathbf{g} \cdot \nabla \xi \, dxdt \\
+ \int_{Q_T} s f_P \xi \, dxdt = \int_{Q_T} (f_P - f_I) \xi \, dxdt. \quad (2.10)
\end{aligned}
$$

3 The finite volume scheme

Now, we want to discretize the problem (2.1)-(2.2). Let $\mathcal{T}$ be a regular and admissible mesh of the domain Ω, constituting of open and convex polygons called control volumes with maximum size (diameter) h.
We let Ω be an open bounded polygonal connected subset of R^3 with boundary $\partial\Omega$. Let $\mathcal{T}$ be an admissible mesh of the domain Ω consisting of open and convex polygons called control volumes with maximum size (diameter) h. For all $K \in \mathcal{T}$, let by x_K denote the center of K, $N(K)$ the set of the neighbors of K i.e. the set of cells of $\mathcal{T}$ which have a common interface with K, by $N_{\text{int}}(K)$ the set of the neighbors of K located in the interior of $\mathcal{T}$, by $N_{\text{ext}}(K)$ the set of edges of K on the boundary $\partial\Omega$.
Furthermore, for all $L \in N_{int}(K)$ denote by $d_{K,L}$ the distance between x_K and x_L, by $\sigma_{K,L}$ the interface between K and L, by $\eta_{K,L}$ the unit normal vector to $\sigma_{K,L}$ outward to K. And for all $\sigma \in N_{\text{ext}}(K)$, denoted by $d_{K,\sigma}$ the distance from x_K to σ.
For all $K \in \mathcal{T}$, we denote by $|K|$ the measure of K. The admissibility of $\mathcal{T}$ implies that $\overline{\Omega} = \cup_{K\in\mathcal{T}} \overline{K}$, $K \cap L = \emptyset$ if $K, L \in \mathcal{T}$ and $K \neq L$, and there exist a finite sequence of points $(x_K)_{K\in\mathcal{T}}$ and the straight line $\overline{x_K x_L}$ is orthogonal to the edge $\sigma_{K,L}$. We also need some regularity on the mesh :

$$\min_{K\in\mathcal{T}, L\in N(K)} \frac{d_{K,L}}{\text{diam}(K)} \geq \alpha$$

for some $\alpha \in \mathbb{R}^+$.
We denote by $H_h(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{T}$. For all $u_h \in H_h(\Omega)$ and for all $K \in \mathcal{T}$, we denote by U_K the constant value of u_h in K. For $(u_h, v_h) \in (H_h(\Omega))^2$, we define the following inner product :

$$\langle u_h, v_h \rangle_{H_h} = l \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (U_L - U_K)(V_L - V_K).$$

In the case of homogeneous Neumann boundary condition, for example $\nabla u \cdot \eta =$

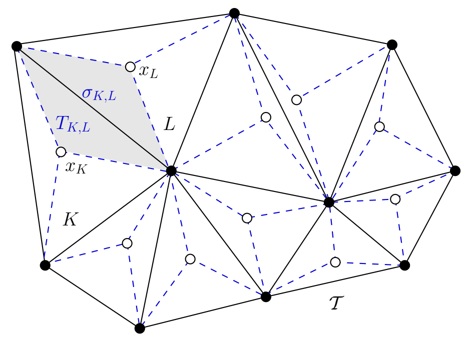

FIGURE 4.1 – Control volumes, centers and diamonds

$\nabla v \cdot \eta = 0$ on $\Gamma_i \subset \partial\Omega$, so we impose $U_L - U_K = V_L - V_K = 0$ if $\sigma_{K,L} \subset \Gamma_i$. And in the case of homogeneous Dirichlet boundary condition $u = v = 0$ on $\Gamma_w \subset \partial\Omega$, so we impose $U_L = V_L = 0$ if $\sigma_{K,L} \subset \Gamma_w$ and $d_{K,L}$ denotes the distance form x_K to $\sigma_{K,L}$, more precisely,

$$\langle u_h, v_h \rangle_{H_h} = l \sum_{K \in \mathcal{T}} \sum_{L \in N_{int}(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (U_L - U_K)(V_L - V_K) + l \sum_{K \in \mathcal{T}} \sum_{\sigma \in N_{ext}(K) \cap \Gamma_w} \frac{|\sigma|}{d_{K,\sigma}} U_K V_K.$$

We define a norm in $H_h(\Omega)$ by

$$\|u_h\|_{H_h(\Omega)} = (\langle u_h, u_h \rangle_{H_h})^{1/2}.$$

Finally, we define $L_h(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{T}$ with the associated norm

$$(u_h, v_h)_{L_h(\Omega)} = \sum_{K \in \mathcal{T}} |K| \, U_K V_K, \qquad \|u_h\|^2_{L_h(\Omega)} = \sum_{K \in \mathcal{T}} |K| \, |U_K|^2 ,$$

for $(u_h, v_h) \in (L_h(\Omega))^2$.

Next, we let $K \in \mathcal{T}$ and $L \in N(K)$ with common vertexes $(a_{\ell,K,L})_{1 \le \ell \le I}$ with $I \in {\scriptscriptstyle N}^\star$. Next let $T_{K,L}$ (respectivley $T^{\text{ext}}_{K,\sigma}$ for $\sigma \in N_{\text{ext}}(K)$) be the open and convex polygon with vertexes (x_K, x_L) (x_K respectively) and $(a_{\ell,K,L})_{1\leq\ell\leq I}$. Observe that

$$\Omega = \cup_{K\in\mathcal{T}}\left(\left(\cup_{L\in N(K)}\overline{T}_{K,L}\right)\cup\left(\cup_{\sigma\in N_{\text{ext}}(K)}\overline{T}^{\text{ext}}_{K,\sigma}\right)\right)$$

The discrete gradient $\nabla_h u_h$ of a constant per control volume function u_h is defined as the constant per diamond $T_{K,L}$ $\mathbb{R}^l$-valued function with values

$$\nabla_h u_h(x) = \begin{cases} l\frac{U_L-U_K}{d_{K,L}}\eta_{K,L} \text{ if } x\in T_{K,L}, \\ l\frac{U_\sigma-U_K}{d_{K,\sigma}}\eta_{K,\sigma} \text{ if } x\in T^{\text{ext}}_{K,\sigma}, \end{cases}$$

Notice that :

• The l-dimensional mesure $|T_{K,L}|$ of $T_{K,L}$ equals to $\frac{1}{l}\left|\sigma_{K,L}\right| d_{K,L}$.

• The semi-norm $\|u_h\|_{H_h}$ coincides with the $L^2(\Omega)$ norm of $\nabla_h u_h$, in fact

$$\begin{aligned}\|\nabla_h u_h\|^2_{L^2(\Omega)} &= \sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\int_{T_{K,L}}|\nabla_h u_h|^2\,dx = l^2\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K,L}|\,\frac{|U_L-U_K|^2}{|d_{K,L}|^2} \\ &= l\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}|U_L-U_K|^2 = \|u_h\|^2_{H_h(\Omega)}\end{aligned}$$

• Let $\vec{F}_{K,L}$ for an arbitrary $\mathbb{R}^l$ vector associated with the interface $\sigma_{K,L}$ satisfying $\vec{F}_{K,L} = \vec{F}_{L,K}$. We denote by $\mathcal{E}_h$ the set of interfaces $\sigma_{K,L}$. Then, a discrete field $(\vec{F}_{K,L})_{\sigma_{K,L}\in\mathcal{E}_h}$ is assimilated to the piecewise constant vector function

$$\vec{F}_h = \sum_{\sigma_{K,L}\in\mathcal{E}_h}\vec{F}_{L,K}\chi_{T_{K,L}}.$$

The discrete divergence of the field $\vec{F}_h$ is defined as the discrete function $w_h = div_h\vec{F}_h$ with the entires

$$div_K\vec{F}_h := \frac{1}{|K|}\sum_{L\in N(K)}|\sigma_{K,L}|\vec{F}_{K,L}\cdot\eta_{K,L}.$$

A key point of the analysis of the two-point finite volume schemes is the following kind of discrete duality property :

Lemma 4.1. *For all discrete function w_h on Ω which is null on $\partial\Omega$, for all discrete field $\vec{F}_h$ on Ω,*

$$\sum_{K\in\mathcal{T}}|K|\,w_K div_K\vec{F}_h = -\sum_{\sigma_{K,L}\in\mathcal{E}_h}|T_{K,L}|\,\nabla_{K,L}w_h\cdot\vec{F}_{K,L}.$$

Démonstration.

$$\begin{aligned}\sum_{K\in\mathcal{T}}|K|\,w_K div_K\vec{F}_h &= \sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|\sigma_{K,L}|\,w_K\vec{F}_{K,L}\cdot\eta_{K,L}\\ &= -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|\sigma_{K,L}|\,(w_L-w_K)\eta_{K,L}\cdot\vec{F}_{K,L}\\ &= -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{1}{l}\,|\sigma_{K,L}|\,d_{K,L}\;l\frac{(w_L-w_K)}{d_{K,L}}\eta_{K,L}\cdot\vec{F}_{K,L}\\ &= -\sum_{\sigma_{K,L}\in\mathcal{E}_h}|T_{K,L}|\,\nabla_{K,L}w_h\cdot\vec{F}_{K,L}.\end{aligned}$$

□

Next, we approximate $M_i(s)\nabla p\cdot\eta_{K,L},\quad(i=1,2.)$ by means of the values s_K,s_L and p_K,p_L that are available in the neighborhood of the interface $\sigma_{K,L}$. To do this, let us use some function G_i of $(a,b,c)\in\mathbb{R}^3$. The numerical convection flux functions G_i, $G_i\in C(\mathbb{R}^3,\mathbb{R})$, satisfies the following properties :

$$\begin{cases}\text{(a) } G_i(\cdot,b,c) \text{ is non-decreasing for all } b,c\in\mathbb{R},\\ \quad\text{and } G_i(a,\cdot,c) \text{ is non-increasing for all } a,c\in\mathbb{R};\\ \text{(b) } G_1(a,a,c)=-M_1(a)\,c \text{ and } G_2(a,a,c)=M_2(a)\,c \text{ for all } a,c\in\mathbb{R};\\ \text{(c) } G_i(a,b,c)=-G_i(b,a,-c) \text{ and there exists } C>0 \text{ such that}\\ \quad |G_i(a,b,c)|\le C\Big(|a|+|b|\Big)|c| \text{ for all } a,b,c\in\mathbb{R}\\ \text{(d) There exists a constant } m_0 \text{ such that}\\ \;(G_2(a,b,c)-G_1(a,b,c))c\ge m_0|c|^2 \text{ for all } a,b,c\in\mathbb{R}.\end{cases}\tag{3.1}$$

Remark 4.1. *Note that the assumptions (a), (b) and (c) are standard and they respectively ensure the maximum principle on saturation, the consistency of the numerical flux, and the conservation of the numerical flux on each interface. Moreover, the last assumption (d) will be used to obtain the L^2 estimate of discrete gradient of the pressure p. Practical examples of numerical convective flux functions can be found in [35].*
In our context, one possibility to construct the numerical flux G_i satisfying (3.1) is to split M_i in the non-decreasing part $M_{i\uparrow}$ and the non-increasing part $M_{i\downarrow}$:

$$M_{i\uparrow}(z):=\int_0^z(M_i'(s))^+\,ds\quad M_{i\downarrow}(z):=-\int_0^z(M_i'(s))^-\,ds.$$

Herein, $s^+=\max(s,0)$ and $s^-=\max(-s,0)$. Then we take

$$G_i(a,b;c)=c^+\Big(M_{i\uparrow}(a)+M_{i\downarrow}(b)\Big)-c^-\Big(M_{i\uparrow}(b)+M_{i\downarrow}(a)\Big),$$

which leads to,

$$\begin{cases} G_1(a,b;c) = -M_1(b)\,c^+ - (-M_1(a))\,c^- \\ G_2(a,b;c) = M_2(b)\,c^+ - M_2(a)\,c^- \end{cases} \tag{3.2}$$

Note that the function $s \mapsto M_1(s)$ is non-decreasing, and the function $s \mapsto M_2(s)$ is non-increasing which lead to the monotony property of the function G_i. Furthermore, depending on the assumption (H2) on the total mobility we have,

$$\Big(G_2(a,b,c) - G_1(a,b,c)\Big)c = M(b){c^+}^2 + M(a){c^-}^2 \geq m_0 c^2. \tag{3.3}$$

The next goal is to discretize the problem (2.1)-(2.2). We denote by $\mathcal{D}$ an admissible discretization of Q_T, which consists of an admissible mesh of Ω, a time step $\Delta t > 0$, and a positive number N chosen as the smallest integer such that $N\Delta t \geq T$. We set,

$$\begin{aligned}
t^n &:= n\Delta t && \text{for } n \in \{0,\dots,N\}\\
dp^{n+1}_{K,L} &:= \frac{|\sigma_{K,L}|}{d_{K,L}}(p^{n+1}_L - p^{n+1}_K) && \text{for } n \in \{0,\dots,N-1\}\\
\rho^{n+1}_{K,L} &:= \frac{1}{p^{n+1}_L - p^{n+1}_K}\int_{p^{n+1}_K}^{p^{n+1}_L} \rho(\zeta)\,d\zeta && \text{for } n \in \{0,\dots,N-1\}\\
f^{n+1}_{P,K} &:= \frac{1}{\Delta t\,|K|}\int_{t^n}^{t^{n+1}}\int_K f_p(x)\,dxdt && \text{for } n \in \{0,\dots,N-1\}\\
f^{n+1}_{I,K} &:= \frac{1}{\Delta t\,|K|}\int_{t^n}^{t^{n+1}}\int_K f_I(x)\,dxdt && \text{for } n \in \{0,\dots,N-1\}\\
\mathbf{g}_{K,L} &:= \int_{K/L}(\mathbf{g}\cdot\eta_{K,L})^+\,d\gamma(x) = \int_{K/L}(\mathbf{g}\cdot\eta_{L,K})^-\,d\gamma(x)
\end{aligned}$$

A finite volume scheme for the discretization of the problem (2.1)-(2.2) is giving by the following set of equations with unknowns $P = (p^{n+1}_K)_{K\in\mathcal{T}}, n \in [0,N]$ and $S = (s^{n+1}_K)_{K\in\mathcal{T}}, n \in [0,N]$, for all $K \in \mathcal{T}$ and $n \in [0,N]$

$$p^0_K = \frac{1}{|K|}\int_K p_0(x)\,dx, \qquad s^0_K = \frac{1}{|K|}\int_K s_0(x)\,dx, \tag{3.4}$$

and

$$\begin{aligned}
&|K|\,\phi_K \frac{\rho(p^{n+1}_K)s^{n+1}_K - \rho(p^n_K)s^n_K}{\Delta t} - \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}\rho^{n+1}_{K,L}(\beta(s^{n+1}_L) - \beta(s^{n+1}_K))\\
&+ \sum_{L\in N(K)} \rho^{n+1}_{K,L} G_1(s^{n+1}_K, s^{n+1}_L; dp^{n+1}_{K,L}) + F^{(n+1)}_{1,K} + |K|\,\rho(p^{n+1}_K)s^{n+1}_K f^{n+1}_{P,K} = 0,
\end{aligned} \tag{3.5}$$

$$|K|\,\phi_K \frac{s_K^{n+1}-s_K^n}{\Delta t} - \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(\beta(s_L^{n+1})-\beta(s_K^{n+1}))$$
$$+\sum_{L\in N(K)} G_2(s_K^{n+1},s_L^{n+1};dp_{K,L}^{n+1}) + F_{2,K}^{(n+1)} + |K|\,(s_K^{n+1}-1)f_{P,K}^{n+1} = -\,|K|\,f_{I,K}^{n+1}, \tag{3.6}$$

where $F_{1,K}^{n+1}$ the approximation of $\int_{\partial K}\rho^2(p^{n+1})M_1(s^{n+1})\mathbf{g}\cdot\eta_{K,L}\,d\gamma(x)$ by an upwind scheme :

$$F_{1,K}^{n+1} = \sum_{L\in N(K)} F_{1,K,L}^{n+1} = \sum_{L\in N(K)} \Big(\rho^2(p_K^{n+1})M_1(s_K^{n+1})\mathbf{g}_{K,L} - \rho^2(p_L^{n+1})M_1(s_L^{n+1})\mathbf{g}_{L,K}\Big), \tag{3.7}$$

and similarly $F_{2,K}^{n+1}$ the approximation of $\int_{\partial K}\rho_2 M_2(s^{n+1})\mathbf{g}\cdot\eta_{K,L}\,d\gamma(x)$ such that

$$F_{2,K}^{(n+1} = \sum_{L\in N(K)} F_{2,K,L}^{(n+1} = \sum_{L\in N(K)} \Big(\rho_2 M_2(s_L^{n+1})\mathbf{g}_{K,L} - \rho_2 M_2(s_K^{n+1})\mathbf{g}_{L,K}\Big). \tag{3.8}$$

Note that the numerical fluxes to approach the gravity terms F_1, F_2 are nondecreasing with respect to s_K and nonincreasing with respect s_L.
We extend the mobility functions $s\mapsto M_1(s)$ and $s\mapsto M_2(s)$ outside $[0,1]$ by continues constant functions as follows, The approximate solutions, $p_{\delta t,h}$, $s_{\delta t,h}$: $\mathbb{R}^+\times\Omega\to\mathbb{R}$ given for all $K\in\mathcal{T}$ and $n\in[0,N]$ by

$$p_{\delta t,h}(t,x) = p_K^{n+1} \text{ and } s_{\delta t,h}(t,x) = s_K^{n+1}, \tag{3.9}$$

for all $x\in K$ and $t\in(n\Delta t,(n+1)\Delta t)$.

The main result of this paper is the following theorem.
Theorem 4.1. *Assume that (H1)-(H5) hold. Let $(p_0,s_0)\in L^2(\Omega,\mathbb{R})\times L^\infty(\Omega,\mathbb{R})$ and $0\le s_0\le 1$ a.e. in Ω. Then there exists an approximate solution $(p_{\delta t,h},s_{\delta t,h})$ to the system (3.5)-(3.6), which converges (up to a subsequence) to (p,s) as $(\delta t,h)\to(0,0)$, where (p,s) is a weak solution to the system (2.1)-(2.2) in the sense of the Definition 4.1.*

4 A priori estimates

We are now concerned with a uniform estimate on the discrete gradient of $\beta(s)$, and on the discrete gradient of the global pressure p.

4.1 Nonnegativity

We aim to prove the following lemma which is a basis to the analysis that we are going to perform.

Lemma 4.2. *Let* $(s_K^0)_{K\in\mathcal{T}} \in [0,1]$. *Then, the solution* $(s_K^n)_{K\in\mathcal{T},n\in\{0,\dots,N\}}$, *of the finite volume scheme* (3.4)-(3.6) *remains in* $[0,1]$.

Démonstration. Let us show by induction in n that for all $K \in \mathcal{T}$, $s_K^n \geq 0$. The claim is true for $n = 0$ and for all $K \in \mathcal{T}$. We argue by induction that for all $K \in \mathcal{T}$, the claim is true up to order n. We consider the control volume K such that $s_K^{n+1} = \min\{s_L^{n+1}\}_{L\in\mathcal{T}}$, and we seek that $s_K^{n+1} \geq 0$.
For the above mentioned purpose, multiply the equation in (3.5) by $-(s_K^{n+1})^-$, we obtain

$$\begin{aligned}
&-|K|\,\phi_K \frac{\rho(p_K^{n+1})s_K^{n+1} - \rho(p_K^n)s_K^n}{\Delta t}(s_K^{n+1})^- \\
&+ \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \rho_{K,L}^{n+1}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))(s_K^{n+1})^- \\
&- \sum_{L\in N(K)} \rho_{K,L}^{n+1} G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})(s_K^{n+1})^- \\
&- F_{1,K}^{(n+1)}(s_K^{n+1})^- - |K|\,\rho(p_K^{n+1})s_K^{n+1} f_{P,K}(s_K^{n+1})^- = 0,
\end{aligned} \tag{4.10}$$

Observe that $\beta(s_L^{n+1}) - \beta(s_K^{n+1}) \geq 0$ (recall that β is nondecreasing). Which implies

$$\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (\beta(s_L^{n+1}) - \beta(s_K^{n+1}))(s_K^{n+1})^- \geq 0. \tag{4.11}$$

The numerical flux G_1 is nonincreasing with respect to s_L^{n+1} (see (a) in (3.1)), and consistence (see (c) in (3.1)), we get

$$\begin{aligned}
G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\,(s_K^{n+1})^- &\leq G_1(s_K^{n+1}, s_K^{n+1}; dp_{K,L}^{n+1})\,(s_K^{n+1})^- \\
&= dp_{K,L}^{n+1}\, M_1(s_K^{n+1})\,(s_K^{n+1})^- = 0.
\end{aligned} \tag{4.12}$$

Using the identity $s_K^{n+1} = (s_K^{n+1})^+ - (s_K^{n+1})^-$, and the mobility M_1 extended by zero on $]-\infty, 0]$, then $M_1(s_K^{n+1})(s_K^{n+1})^- = 0$ and

$$\begin{aligned}
&- F_{1,K}^{(n+1)}(s_K^{n+1})^- - |K|\,\rho(p_K^{n+1})s_K^{n+1} f_{P,K}^{n+1}(s_K^{n+1})^- \\
&= \sum_{L\in N(K)} \rho^2(p_L^{n+1})M_1(s_L^{n+1})\mathbf{g}_{L,K}(s_K^{n+1})^- + |K|\,\rho(p_K^{n+1})f_{P,K}|(s_K^{n+1})^-|^2 \geq 0.
\end{aligned} \tag{4.13}$$

Then, we deduce from (4.10) that

$$|K|\,\phi_K \frac{\rho(p_K^{n+1})|(s_K^{n+1})^-|^2 + \rho(p_K^n) s_K^n (s_K^{n+1})^-}{\Delta t} \leq 0,$$

and from the nonnegativity of s_K^n, we obtain $(s_K^{n+1})^- = 0$. This implies that $s_K^{n+1} \geq 0$ and

$$0 \leq s_K^{n+1} \leq s_L^{n+1} \text{ for all } n \in [0, N-1] \text{ and } L \in \mathcal{T}.$$

To prove that $s_K^{n+1} \leq 1$ for all $n \in [0, N-1]$ and $K \in \mathcal{T}$. We argue by induction that for all $K \in \mathcal{T}$, $s_K^n \leq 1$. Let the control volume K such that $s_K^{n+1} = \max\{s_L^{n+1}\}_{L\in\mathcal{T}}$, and let us show that $s_K^{n+1} \leq 1$.
For the mentioned claim, we multiply the equation in (3.6) by $(s_K^{n+1} - 1)^+$,

$$\begin{aligned}
|K|\,\phi_K \frac{s_K^{n+1} - s_K^n}{\Delta t}(s_K^{n+1} - 1)^+ - \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))(s_K^{n+1} - 1)^+ \\
+ \sum_{L\in N(K)} G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})(s_K^{n+1} - 1)^+ + F_{2,K}^{(n+1)}(s_K^{n+1} - 1)^+ \\
+ |K|\,(s_K^{n+1} - 1) f_{P,K}(s_K^{n+1} - 1)^+ = -\,|K|\, f_{I,K}^{n+1}(s_K^{n+1} - 1)^+
\end{aligned} \tag{4.14}$$

Since β is nondecreasing, we get $\beta(s_L^{n+1}) - \beta(s_K^{n+1}) \leq 0$. This implies

$$-\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))(s_K^{n+1} - 1)^+ \geq 0. \tag{4.15}$$

Next, we use the fact that the numerical flux G_2 is nondecreasing with respect to s_K^{n+1} and consistence (see (b) and (c) in (3.1) to deduce

$$\begin{aligned}
G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\,(s_K^{n+1} - 1)^+ &\geq G_2(s_K^{n+1}, s_K^{n+1}; dp_{K,L}^{n+1})\,(s_K^{n+1} - 1)^+ \\
&= dp_{K,L}^{n+1}\, M_2(s_K^{n+1})\,(s_K^{n+1} - 1)^+ = 0,
\end{aligned} \tag{4.16}$$

now, we rely on the extension of the mobility M_2 by zero on $[1, \infty[$, thus $M_2(s_K^{n+1})$ $(s_K^{n+1} - 1)^+ = 0$, to deduce

$$F_{2,K}^{(n+1)}(s_K^{n+1} - 1)^+ = \sum_{L\in N(K)} \rho_2 M_2(s_L^{n+1}) \mathbf{g}_{K,L}(s_K^{n+1} - 1)^+ \geq 0 \tag{4.17}$$

It is clear that the production source term in the left hand side of (4.14) is nonnegative and the injection source term on the right hand side is nonpositive.

Using the above estimates to deduce from (4.14) that,

$$|K|\,\phi_K \frac{s_K^{n+1}-s_K^n}{\Delta t}(s_K^{n+1}-1)^+ = \\ \frac{|K|\,\phi_K}{\Delta t}\Big((s_K^{n+1}-1)(s_K^{n+1}-1)^+ - (s_K^n-1)(s_K^{n+1}-1)^+\Big) \le 0 \quad (4.18)$$

Using again the identity $(s_K^{n+1}-1) = (s_K^{n+1}-1)^+ - (s_K^{n+1}-1)^-$, and that $s_K^n \le 1$ to deduce from (4.18) that $(s_K^{n+1}-1)^+ = 0$. Consequently, we obtain

$$s_L^{n+1} \le s_K^{n+1} \le 1 \text{ for all } n \in [0, N-1] \text{ and } L \in \mathcal{T}.$$

□

4.2 Discrete a priori estimates

Let us recall the following two lemmas : :

Lemma 4.3. *(Discrete Poincaré inequality)[35]*
Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, $\mathcal{T}$ an admissible finite volume mesh in the sense given in the section 3, and let u be a function which is constant on each cell $K \in \mathcal{T}$, that is, $u(x) = u_K$ if $x \in K$, then

$$\|u\|_{L^2(\Omega)} \le diam(\Omega)\,\|u\|_{H_h(\Omega)}\,,$$

where $\|\cdot\|_{H_h(\Omega)}$ is the discrete H_0^1 norm.

Remark 4.2. *(Dirichlet condition on part of the boundary) This lemma gives a discrete Poincaré inequality for Dirichlet boundary conditions on the boundary $\partial\Omega$. In the case of Dirichlet condition on part of the boundary only, it is still possible to prove a discrete Poincaré inequality provided that the polygonal bounded open set Ω is connected.*

Lemma 4.4. *(Discrete integration by parts formula) Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $\mathcal{T}$ an admissible finite volume mesh in the sense given in the subsection 3. Let $F_{K/L}$, $K \in \mathcal{T}$ and $L \in N(K)$ be a value in $\mathbb{R}$ depends on K and L such that $F_{K/L} = -F_{L/K}$, and let φ be a function which is constant on each cell $K \in \mathcal{T}$, that is, $\varphi(x) = \varphi_K$ if $x \in K$, then*

$$\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} F_{K/L}\varphi_K = -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} F_{K/L}(\varphi_L - \varphi_K) \quad (4.19)$$

Consequently, if $F_{K/L} = a_{K/L}(b_L - b_K)$*, with* $a_{K/L} = a_{L/K}$*, then*

$$\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} a_{K/L}(b_L - b_K)\varphi_K = -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} a_{K/L}(b_L - b_K)(\varphi_L - \varphi_K) \quad (4.20)$$

Démonstration. The sum $\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}$ can be reorganized by edge. In fact, on each edge $\sigma_{K,L}$ between the mesh K and L, there are two contributions : from K to L named $F_{K,L}\varphi_K$ and from L to K named $F_{L,K}\varphi_L$, then

$$\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} F_{K/L}\varphi_K = \sum_{\sigma_{K,L}} (F_{K/L}\varphi_K + F_{L/K}\varphi_L) \quad (4.21)$$

Using now the fact that $F_{K/L}$ is antisymmetric, then we have

$$\begin{aligned}\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} F_{K/L}\varphi_K &= \sum_{\sigma_{K,L}} F_{K/L}(\varphi_K - \varphi_L)\\ &= \frac{1}{2}\sum_{\sigma_{K,L}}\Big(F_{K/L}(\varphi_K - \varphi_L) + F_{L/K}(\varphi_L - \varphi_K)\Big).\end{aligned} \quad (4.22)$$

Finally, reorganise the last summation on edge by mesh, we obtain exaclty (4.19). The equality (4.20) is a direct consequence of (4.19) $F_{K/L} = a_{K/L}(b_L - b_K) = a_{K/L}(b_K - b_L) = -F_{L/K}$. $\square$

We derive in the next proposition, the main uniform estimates on the discrete gradient of the capillary term $\beta(s)$ and the discrete gradient of the global pressure p.

Proposition 4.1. *Let* $(p_K^n, s_K^n)_{K\in\mathcal{T}, n\in\{0,\dots,N\}}$*, be a solution of the finite volume scheme* (3.5)-(3.6)*. Then, there exist a constant* $C > 0$*, depending on* Ω*,* T*,* s_0*,* p_0 *and* α *such that*

$$\begin{aligned}&\sum_{K\in\mathcal{T}} |K|\, s_K^N \mathcal{H}(p_K^N) - \sum_{K\in\mathcal{T}} |K|\, s_K^0 \mathcal{H}(p_K^0)\\ &+ \frac{c_1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}\left|p_K^{n+1} - p_L^{n+1}\right|^2 \le C\end{aligned} \quad (4.23)$$

and

$$\begin{aligned}&\sum_{K\in\mathcal{T}} |K|\, B(s_K^N) - \sum_{K\in\mathcal{T}} |K|\, B(s_K^0)\\ &+ \frac{1}{4}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}\left|\beta(s_K^{n+1}) - \beta(s_L^{n+1})\right|^2 \le C\end{aligned} \quad (4.24)$$

where $B'(s) = \beta(s)$*, and* $\mathcal{H}(p) = g(p) + \rho(p)p$ *with* $g'(p) = -\rho(p)$*.*

Démonstration. To prove the estimate (4.23), we multiply the gas discrete equation (3.5) and the water discrete equation (3.6) respectively by p_K^{n+1}, $g(p_K^{n+1}) = \mathcal{H}(p_K^{n+1}) - \rho(p_K^{n+1})p_K^{n+1}$ and adding them, then summing the resulting equation over K and n, and this yields to,

$$E_1 + E_2 + E_3 + E_4 + E_5 = 0 \tag{4.25}$$

where

$$\begin{aligned}
E_1 &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|\, \phi_K \Big((\rho(p_K^{n+1})s_K^{n+1} - \rho(p_K^n)s_K^n)\; p_K^{n+1} + (s_K^{n+1} - s_K^n)\; g(p_K^{n+1}) \Big), \\
E_2 &= -\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \Big(\rho_{K,L}^{n+1}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))\; p_K^{n+1} + \\
&\qquad\qquad (\beta(s_L^{n+1}) - \beta(s_K^{n+1}))\; g(p_K^{n+1}) \Big), \\
E_3 &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \Big(\rho_{K,L}^{n+1} G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\; p_K^{n+1} + \\
&\qquad\qquad G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\; g(p_K^{n+1}) \Big), \\
E_4 &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \Big(F_{1,K,L}^{(n+1)}\; p_K^{n+1} + F_{2,K,L}^{(n+1)}\; g(p_K^{n+1}) \Big), \\
E_5 &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} |K| \Big(\rho(p_K^{n+1})s_K^{n+1} f_{P,K}^{n+1}\; p_K^{n+1} + (s_K^{n+1} - 1) f_{P,K}^{n+1}\; g(p_K^{n+1}) + \\
&\qquad\qquad f_{I,K}^{n+1}\; g(p_K^{n+1}) \Big).
\end{aligned}$$

To handle the first term of the equality (4.25), let us prove that : for all $s \geq 0$ and $s^\star \geq 0$,

$$\Big(\rho(p)s - \rho(p^\star)s^\star\Big)p + (s - s^\star)(\mathcal{H}(p) - \rho(p)p) \geq \mathcal{H}(p)s - \mathcal{H}(p^\star)s^\star. \tag{4.26}$$

Indeed, denote $g(p) = \mathcal{H}(p) - \rho(p)p$ then $g'(p) = -\rho(p)$,

$$\begin{aligned}
&\Big(\rho(p)s - \rho(p^\star)s^\star\Big)p + (s - s^\star)(\mathcal{H}(p) - \rho(p)p) \\
&= s(\mathcal{H}(p) - s^\star(\rho(p^\star)p + g(p)) = s\mathcal{H}(p) - s^\star\mathcal{H}(p^\star) + s^\star\Big(\mathcal{H}(p^\star) - \rho(p^\star)p - g(p)\Big).
\end{aligned}$$

We have to show that

$$\mathcal{H}(p^\star) - \rho(p^\star)p - g(p) \geq 0.$$

We expand this quantity as follows,

$$\mathcal{H}(p^\star) - \rho(p^\star)p - g(p) = g(p^\star) + \rho(p^\star)(p^\star - p) - g(p) = g(p^\star) - g(p) - g'(p^\star)(p^\star - p),$$

as the function g is concave $(g''(p) = -\rho'(p) \le 0)$, we get

$$g(p) \le g(p^\star) + g'(p^\star)(p - p^\star).$$

So, (4.26) is established, and this yields to

$$\sum_{K\in\mathcal{T}} |K|\, s_K^N \mathcal{H}(p_K^N) - \sum_{K\in\mathcal{T}} |K|\, s_K^0 \mathcal{H}(p_K^0) \le E_1 \tag{4.27}$$

Integrating by parts, see *lemma* 4.4, we obtain

$$E_2 = \frac{1}{2}\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \Big(\beta(s_L^{n+1}) - \beta(s_K^{n+1})\Big)$$
$$\Big(\rho_{K,L}^{n+1}(p_L^{n+1} - p_K^{n+1}) + (g(p_L^{n+1}) - g(p_K^{n+1}))\Big).$$

Due to the correct choice of the density of the gas on each interface,

$$\rho_{K,L}^{n+1} = \frac{(g(p_L^{n+1}) - g(p_K^{n+1}))}{(p_K^{n+1} - p_L^{n+1})}$$

we succeed to obtain,

$$E_2 = 0. \tag{4.28}$$

The choice of the density on the interfaces is the key point to vanish the dissipative term on saturation and obtain a uniform estimate on the discrete gradient of pressure p.
Using the fact that the numerical fluxes G_1 and G_2 are conservative in the sense of (c) in (3.1), we can apply *lemma* 4.4 and we obtain

$$E_3 = \frac{1}{2}\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \rho_{K,L}^{n+1}\Big(G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})$$
$$- G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\Big)\Big(p_L^{n+1} - p_K^{n+1}\Big),$$

Recall that inequality (3.3),

$$\Big(G_2(a,b,c) - G_1(a,b,c)\Big)c = M(b){c^+}^2 + M(a){c^-}^2 \ge m_0 c^2,$$

this with the hypothesis (H 5) allow us to deduce that,

$$m_0\rho_m \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \left|p_K^{n+1} - p_L^{n+1}\right|^2 \le E_3. \tag{4.29}$$

To handle the other terms of the equality (4.25), firstly let us remark that the numerical flux satisfies $F^{n+1}_{1,K,L} = -F^{n+1}_{1,L,K}$ and $F^{n+1}_{2,K,L} = -F^{n+1}_{2,L,K}$, so we integrate by parts and we obtain

$$E_4 = \frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|\sigma_{K,L}|\Big(F^{(n+1)}_{1,K,L}(p^{n+1}_K - p^{n+1}_L) + F^{(n+1)}_{2,K,L}(g(p^{n+1}_K) - g(p^{n+1}_L)\Big),$$

use now the fact that the mobilities and densities are bounded from (H2)-(H5), and the map g is uniformly Lipschitz, we have, there exists a positive constant independent of Δt and h such that

$$|E_4| \leq C_1\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|\sigma_{K,L}||p^{n+1}_K - p^{n+1}_L|.$$

From the following inequality $|\sigma_{K,L}| = (|\sigma_{K,L}|d_{K,L})^{\frac{1}{2}}\frac{|\sigma_{K,L}|^{\frac{1}{2}}}{d^{\frac{1}{2}}_{K,L}}$, and apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}|E_4| \leq & C_1\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|\sigma_{K,L}|d_{K,L}\\ &+\frac{m_0\rho_m}{4}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\left|p^{n+1}_K - p^{n+1}_L\right|^2\\ \leq & C_1T|\Omega| + \frac{m_0\rho_m}{4}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\left|p^{n+1}_K - p^{n+1}_L\right|^2.\end{aligned}$$

The last term will be absorbed by the dissipative term on global pressure from the estimate (4.29).
In order to estimate E_5, using again the fact that the densities are bounded and the map g is sublinear (i.e.$|g(p)| \leq C|p|$), we have

$$|E_5| \leq C_1\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}|K|\left(f_{P,K} + f^{n+1}_{I,K}\right)\left|p^{n+1}_K\right|$$

we apply Holder inequality to deduce,

$$\begin{aligned}|E_5| \leq & C_1\Big(\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}|K|\left|f_{P,K} + f^{n+1}_{I,K}\right|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}|K|\left|p^{n+1}_K\right|^2\Big)^{\frac{1}{2}}\\ \leq & C_1(\|f_P + f_I\|_{L^2(Q_T)}\Big(\sum_{n=0}^{N-1}\Delta t\left\|p^{n+1}_h\right\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}\end{aligned}$$

Now, from the discrete Poincaré inegqality lemma 4.3, leads to,

$$|E_5| \leq C_2\Big(\sum_{n=0}^{N-1} \Delta t \left\|p_h^{n+1}\right\|_{H_h}^2\Big)^{\frac{1}{2}},$$

where C_2 is a constant depends only on $\|f_P + f_I\|_{L^2(Q_T)}$. Finally, under the assumption (H3) on the source terms and as an application of Young's inequality ($a \cdot b \leq \eta a^2 + \frac{b^2}{4\eta}$.), we get

$$|E_5| \leq C3 + \frac{m_0 \rho_m}{4} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \left|p_K^{n+1} - p_L^{n+1}\right|^2 \tag{4.30}$$

this estimate (4.30), with (4.27), (4.28) and (4.29) achieve the proof of (4.23). To prove the estimate (4.24), we multiply the water discrete equation in (3.6) by $\beta(s_K^{n+1})$ then summing the resulting equation over K and n, and this yields to

$$J_1 + J_2 + J_3 = 0 \tag{4.31}$$

where

$$\begin{aligned}
J_1 &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K|\, \phi_K (s_K^{n+1} - s_K^n) \cdot \beta(s_K^{n+1}), \\
J_2 &= -\sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (\beta(s_L^{n+1}) - \beta(s_K^{n+1})) \cdot \beta(s_K^{n+1}), \\
J_3 &= \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}) \cdot \beta(s_K^{n+1}) \\
&\quad + \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} F_{2,K,L}^{(n+1)} \cdot \beta(s_K^{n+1}) \\
&\quad + \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} |K| \Big((s_K^{n+1} - 1) f_{P,K}^{n+1} + f_{I,K}^{n+1} \Big) \cdot \beta(s_K^{n+1}).
\end{aligned}$$

Let $B(s) = \displaystyle\int_0^s \beta(r)\, dr$. From the convexity of $B(s)$ (recall that $\beta''(s) = a(s) \geq 0$), we obtain

$$\begin{aligned}
J_1 &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K|\, (s_K^{n+1} - s_K^n) \beta(s_K^{n+1}) \\
&\geq \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K|\, (B(s_K^{n+1}) - B(s_K^n)) \\
&= \sum_{K \in \mathcal{T}} |K| \Big(B(s_K^N) - B(s_K^0) \Big)
\end{aligned} \tag{4.32}$$

Applying *lemma* 4.4, we obtain

$$\begin{aligned} J_2 &= \frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}(\beta(s_L^{n+1})-\beta(s_K^{n+1}))\cdot(\beta(s_L^{n+1})-\beta(s_K^{n+1})) \\ &= \frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}\rho_{K,L}^{n+1}\left|\beta(s_L^{n+1})-\beta(s_K^{n+1})\right|^2. \end{aligned} \tag{4.33}$$

The other terms in the equality (4.31) can be treated as (4.30), using Holder's and Young's inequalities with the help of *lemma* 4.3 and the assumptions on mobilities (H2), source terms (H3) and densities (H5) to get,

$$|J_3| \le C + \frac{1}{4}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}\left|\beta(s_K^{n+1})-\beta(s_L^{n+1})\right|^2 \tag{4.34}$$

Now, collecting (4.32), (4.33) and (4.34) we obtain,

$$\begin{aligned} &\sum_{K\in\mathcal{T}}|K|\,B(s_K^N) - \sum_{K\in\mathcal{T}}|K|\,B(s_K^0) \\ &+\frac{1}{4}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}}\left|\beta(s_K^{n+1})-\beta(s_L^{n+1})\right|^2 \le C, \end{aligned}$$

for some constant $C \ge 0$. This concludes the proof of proposition 4.1. □

5 Existence of the finite volume scheme

The existence of a solution to the finite volume scheme will be obtained with the help of the following lemma proved in [57] and [65].

Lemma 4.5. *Let $\mathcal{A}$ be a finite dimensional Hilbert space with scalar product $[\cdot,\cdot]$ and norm $\|\cdot\|$, and let $\mathcal{P}$ be a continuous mapping from $\mathcal{A}$ into itself such that*

$$[\mathcal{P}(\xi),\xi] > 0 \textit{ for } \|\xi\| = r > 0.$$

Then there exists $\xi \in \mathcal{A}$ with $\|\xi\| \le r$ such that

$$\mathcal{P}(\xi) = 0.$$

The existence for the finite volume scheme is given in

Proposition 4.2. *Let $\mathcal{D}$ be an admissible discretization of Q_T. Then the problem (3.5)-(3.6) admits at least one solution $(p_K^n, s_K^n)_{(K,n)\in\Omega_R\times\{0,\ldots,N\}}$.*

Démonstration. At the beginning of the proof, we set the following notations ;

$$\begin{aligned}\mathcal{M} &:= Card(\mathcal{T})\\ s_{\mathcal{M}} &:= \{s_K^{n+1}\}_{K\in\mathcal{T}} \in \mathbb{R}^{\mathcal{M}},\\ p_{\mathcal{M}} &:= \{p_K^{n+1}\}_{K\in\mathcal{T}} \in \mathbb{R}^{\mathcal{M}}\end{aligned}$$

We define the map $\mathcal{T}_h : \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}} \longrightarrow \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$,

$$\mathcal{T}_h(s_{\mathcal{M}}, p_{\mathcal{M}}) = (\{\mathcal{T}_{1,K}\}_{K\in\mathcal{T}}, \{\mathcal{T}_{2,K}\}_{K\in\mathcal{T}}) \text{ where,}$$

$$\begin{aligned}\mathcal{T}_{1,K} = |K|\,\phi_K \frac{\rho(p_K^{n+1})s_K^{n+1} - \rho(p_K^n)s_K^n}{\Delta t} &- \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \rho_{K,L}^{n+1}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))\\ &+ \sum_{L\in N(K)} \rho_{K,L}^{n+1} G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}) + F_{1,K,L}^{(n+1)} + |K|\,\rho(p_K^{n+1})s_K^{n+1} f_{P,K}^{n+1},\end{aligned} \tag{5.35}$$

$$\begin{aligned}\mathcal{T}_{2,K} = |K|\,\phi_K \frac{s_K^{n+1} - s_K^n}{\Delta t} &- \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (\beta(s_L^{n+1}) - \beta(s_K^{n+1}))\\ &+ \sum_{L\in N(K)} G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}) + F_{2,K,L}^{(n+1)} + |K|\,(s_K^{n+1} - 1) f_{P,K}^{n+1} + |K|\, f_{I,K}^{n+1}.\end{aligned} \tag{5.36}$$

Note that $\mathcal{T}_h$ is well defined as a continues function. Also we define the following homeomorphism $\mathcal{F} : \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}} \mapsto \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{M}}$ such that,

$$\mathcal{F}(p_{\mathcal{M}}, s_{\mathcal{M}}) = (p_{\mathcal{M}}, v_{\mathcal{M}})$$

where, $v_{\mathcal{M}} = \{g(p_K^{n+1}) + \beta(s_K^{n+1})\}_{K\in\mathcal{T}}$.
Now let us consider the following continues mapping $\mathcal{P}_h$ defined as

$$\begin{aligned}\mathcal{P}_h(p_{\mathcal{M}}, v_{\mathcal{M}}) &= \mathcal{T}_h \circ \mathcal{F}^{-1}(p_{\mathcal{M}}, v_{\mathcal{M}})\\ &= \mathcal{T}_h(s_{\mathcal{M}}, p_{\mathcal{M}}).\end{aligned}$$

Our goal now is to show that,

$$[\mathcal{P}_h(p_{\mathcal{M}}, v_{\mathcal{M}}), (p_{\mathcal{M}}, v_{\mathcal{M}})] > 0, \quad \text{for } \|(p_{\mathcal{M}}, v_{\mathcal{M}})\|_{\mathbb{R}^{2\mathcal{M}}} = r > 0, \tag{5.37}$$

and for a sufficiently large r.

We observe that

$$
\begin{aligned}
[\mathcal{P}_h(p_\mathcal{M}, v_\mathcal{M}), (p_\mathcal{M}, v_\mathcal{M})] \geq & \frac{1}{\Delta t}\sum_{K\in\mathcal{T}} |K|\, s_K^{n+1} H(p_K^{n+1}) - \frac{1}{\Delta t}\sum_{K\in\mathcal{T}} |K|\, s_K^{n} H(p_K^{n}) \\
& + \frac{1}{\Delta t}\sum_{K\in\mathcal{T}} |K|\, B(s_K^{n+1}) - \frac{1}{\Delta t}\sum_{K\in\mathcal{T}} B(s_K^{n}) \\
& + C(m_0,\rho_m)\left\| p_h^{n+1}\right\|^2_{H_h(\Omega)} + 1/2 \left\|\beta(s_h^{n+1})\right\|^2_{H_h(\Omega)} - C,
\end{aligned}
$$

for some constants $C(m_0,\rho_m), C>0$. This implies that

$$
\begin{aligned}
[\mathcal{P}_h(p_\mathcal{M}, v_\mathcal{M}), (p_\mathcal{M}, v_\mathcal{M})] \geq & -\frac{1}{\Delta t}\sum_{K\in\mathcal{T}} |K|\, s_K^{n} H(p_K^{n}) - \frac{1}{\Delta t}\sum_{K\in\mathcal{T}} B(s_K^{n}) \\
& + C(\left\| p_h^{n+1}\right\|^2_{H_h(\Omega)} + \left\|\beta(s_h^{n+1})\right\|^2_{H_h(\Omega)}) - C',
\end{aligned}
\tag{5.38}
$$

for some constants $C, C'>0$. Finally using the fact that g is a Lipschitz function, then there exists a constant $C>0$ such that

$$
\left\|(\{p_K^{n+1}\}_{K\in\mathcal{T}}, \{g(p_K^{n+1})+\beta(s_K^{n+1})\}_{K\in\mathcal{T}})\right\|_{\mathbb{R}^{2\mathcal{M}}} \leq C(\left\|\beta(s_h^{n+1})\right\|_{H_h(\Omega)}) + \left\|p_h^{n+1}\right\|_{H_h(\Omega)}.
$$

Using this to deduce from (5.38) that (5.37) holds for r large enough. Hence, we obtain the existence of at least one solution to the scheme (3.5)-(3.6). □

6 Space and time translation estimates

In this section we derive estimates on differences of space and time translates of the function $\phi_h\rho(p_h)s_hB(s_h)$ which imply that the sequence $\phi_h\rho(p_h)s_hB(s_h)$ is relatively compact in $L^1(Q_T)$.
We replace the study of discrete functions $U^h = \phi_h\rho(p_h)s_hB(s_h)$ (constant per cylinder $Q_K^n := (t^n, t^{n+1})] \times K$) by the study of functions $\bar{U}^h = \phi_h\rho(\bar{p}_h)\bar{s}_hB(\bar{s}_h)$ piecewise continuous in t for all x, constant in x for all volume K, defined as

$$
\bar{U}^h(t,x) = \sum_{n=0}^{N_h}\sum_{K\in\mathcal{T}_h}\frac{1}{\Delta t}\Big(\ (t-n\Delta t)U_K^{n+1}\ +\ ((n+1)\Delta t - t)U_K^n\ \Big)\ \mathbb{1}_{Q_K^n}(t,x).
$$

For a given discrete field $\vec{\mathcal{F}}_h := \sum_{\sigma_{K,L}}\vec{\mathcal{F}}_{K,L}\mathbb{1}_{T_{K,L}}$, its discrete divergence is defined as a discrete function with entries on each control volume K;

$$
\operatorname{div}_K\vec{\mathcal{F}}_h := \frac{1}{|K|}\sum_{L\in N(K)}\sigma_{K,L}\vec{\mathcal{F}}_{K,L}\cdot\eta_{K,L}
$$

Observe that we can write the discrete scheme (3.5)-(3.6) in the following from :

$$\phi_h \frac{\rho(p_h^{n+1})s_h^{n+1} - \rho(p_h^n)s_h^n}{\Delta t} = \mathrm{div}_h \vec{\mathcal{F}}_{1,h}^{n+1} + f_{1,h}^{n+1}$$
$$\phi_h \frac{s_h^{n+1} - s_h^n}{\Delta t} = \mathrm{div}_h \vec{\mathcal{F}}_{2,h}^{n+1} + f_{2,h}^{n+1}.$$

where,

$$\vec{\mathcal{F}}_{1,h}^{n+1} := \sum_{\sigma_{K,L}} \Big(\frac{\rho_{K,L}^{n+1}}{d_{K,L}} (\beta(s_L^{n+1}) - \beta(s_K^{n+1})) - \frac{\rho_{K,L}^{n+1}}{\sigma_{K,L}} G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})$$
$$- \frac{1}{\sigma_{K,L}} (\rho^2(p_K^{n+1}) M_1(s_K^{n+1}) \mathbf{g}_{K,L} - \rho^2(p_L^{n+1}) M_1(s_L^{n+1}) \mathbf{g}_{L,K}) \Big) \cdot \eta_{K,L} 1\!\!1_{T_{K,L}}$$

$$\vec{\mathcal{F}}_{2,h}^{n+1} := \sum_{\sigma_{K,L}} \Big(\frac{1}{d_{K,L}} (\beta(s_L^{n+1}) - \beta(s_K^{n+1})) - \frac{1}{\sigma_{K,L}} G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})$$
$$- \frac{1}{\sigma_{K,L}} (\rho_2 M_2(s_L^{n+1}) \mathbf{g}_{K,L} - \rho_2 M_2(s_K^{n+1}) \mathbf{g}_{L,K}) \Big) \cdot \eta_{K,L} 1\!\!1_{T_{K,L}}$$

$$f_{1,h}^{n+1} := \rho(p_K^{n+1}) s_K^{n+1} f_{P,K}^{n+1}$$
$$f_{2,h}^{n+1} := -(s_K^{n+1} - 1) f_{P,K}^{n+1} - |K| \, f_{I,K}^{n+1}.$$

We also extend $\bar{U}^h$ by the constant in time value U^{h,N_h+1} on $[\Delta t(N_h+1), +\infty)$; as to $\vec{\mathcal{F}}_{1,h}$, $\vec{\mathcal{F}}_{2,h}$, $f_{1,h}$ and $f_{2,h}$, they are extended by zero values for $t > \Delta t(N_h+1)$. The above definitions permit us to rewrite the equations (3.5)-(3.6) under the form

$$\begin{aligned} \partial_t \phi_h \rho(\bar{p}_h) \bar{s}_h &= \mathrm{div}_h \vec{\mathcal{F}}_{1,h} + f_{1,h} \\ \partial_t \phi_h \bar{s}_h &= \mathrm{div}_h \vec{\mathcal{F}}_{2,h} + f_{2,h}, \end{aligned} \tag{6.39}$$

where $\vec{\mathcal{F}}_{1,h}$, $\vec{\mathcal{F}}_{2,h}$, $f_{1,h}$ and $f_{2,h}$ are respectively the discrete functions of values $\vec{\mathcal{F}}_{1,h}^{n+1}$, $\vec{\mathcal{F}}_{2,h}^{n+1}$, $f_{1,h}^{n+1}$ and $f_{2,h}^{n+1}$ on each interval $]t^n, t^{n+1}]$.
These equations are satisfied in $W^{1,1}(\mathbb{R}^+)$ in time, for a.e. $x \in \Omega$.
Lemma 4.6. *There exists positive a constant $C > 0$ depending on Ω, T, u_0 and v_0 such that*

$$\iint_{\Omega' \times (0,T)} \left| \bar{U}(t, x+y) - \bar{U}(t,x) \right| \, dx \, dt \le C \, |y| \, (|y| + 2h), \tag{6.40}$$

for all $y \in \mathbb{R}^3$ with $\Omega' = \{x \in \Omega, \, [x, x+y] \subset \Omega\}$, and

$$\iint_{\Omega \times (0,T-\tau)} \left| \bar{U}(t+\tau, x) - \bar{U}(t,x) \right| \, dx \, dt \le C(\tau + \Delta t), \tag{6.41}$$

for all $\tau \in (0,T)$.

Démonstration. The proof is similar to that found in, e.g, [33].
Proof of (6.40). First to simplify the notation, we write

$$\sum_{\sigma_{K,L}} \quad \text{instead of} \quad \sum_{\{(K,L)\in\mathcal{T}^2,\, K\neq L,\, m(\sigma_{K,L})\neq 0\}}.$$

Let $y \in \mathbb{R}^3$, $x \in \Omega'$, and $L \in N(K)$. We set

$$\beta_{\sigma_{K,L}} = \begin{cases} 1, & \text{if the line segment } [x, x+y] \text{ intersects } \sigma_{K,L},\ K \text{ and } L, \\ 0, & \text{otherwise.} \end{cases}$$

Next, the value $c_{\sigma_{K,L}}$ is defined by $c_{\sigma_{K,L}} = \dfrac{y}{|y|} \cdot \eta_{K,L}$ with $c_{\sigma_{K,L}} > 0$. We observe that (see for more details [35])

$$\begin{aligned} &\int_{\Omega'} \beta_{\sigma_{K,L}}(x)\,dx \le m(\sigma_{K,L})\,|y|\,c_{\sigma_{K,L}}, \\ &\sum_{\sigma_{K,L}} \beta_{\sigma_{K,L}}(x) c_{\sigma_{K,L}} d_{K,L} \le |y| + 2h. \end{aligned} \tag{6.42}$$

With this and an application of the Cauchy-Schwarz inequality leads to

$$\begin{aligned} &\iint_{(0,T)\times\Omega'} \left|U^h(t,x+y) - U^h(t,x)\right|^2 dx \\ &\le \sum_{\sigma_{K,L}} \beta_{\sigma_{K,L}}(x) c_{\sigma_{K,L}} d_{K,L} \sum_{n=0}^{N-1} \Delta t \sum_{\sigma_{K,L}} \frac{\left|U_L^{n+1} - U_K^{n+1}\right|^2}{c_{\sigma_{K,L}} d_{K,L}} \int_{\Omega'} \beta_{\sigma_{K,L}}(x)\,dx \\ &\le (|y| + 2h) \sum_{n=0}^{N-1} \Delta t \sum_{\sigma_{K,L}} \frac{\left|U_L^{n+1} - U_K^{n+1}\right|^2}{c_{\sigma_{K,L}} d_{K,L}} \int_{\Omega'} \beta_{\sigma_{K,L}}(x)\,dx \\ &\le |y|\,(|y| + 2h) \sum_{n=0}^{N-1} \Delta t \sum_{\sigma_{K,L}} \frac{\left|\phi\rho(p_L^{n+1}) s_L^{n+1} B(s_L^{n+1}) - \phi\rho(p_K^{n+1}) s_K^{n+1} B(s_K^{n+1})\right|^2}{c_{\sigma_{K,L}} d_{K,L}} \\ &\le C\,|y|\,(|y| + 2h) \sum_{n=0}^{N-1} \Delta t \sum_{\sigma_{K,L}} \frac{\left(\left|p_L^{n+1} - p_L^{n+1}\right|^2 + \left|\beta(s_L^{n+1}) - \beta(s_K^{n+1})\right|^2\right)}{c_{\sigma_{K,L}} d_{K,L}}, \end{aligned} \tag{6.43}$$

for some constant $C > 0$. In addition, we have

$$\int_0^{+\infty}\!\!\int_{\Omega'} |\bar U^h(t,x+dx) - \bar U^h(t,x)|\,dxdt \leq 2\int_0^T\!\!\int_{\Omega'} |U^h(t,x+dx) - U^h(t,x)|\,dxdt + 2\Delta t_h \int_{\Omega'_\Delta} |U_0^h(x)|\,dx,$$

where $U_0 = \rho(p_0)s_0B(s_0)$ and $\Omega'_\Delta = \{x \in \Omega,\ \mathrm{dist}\,(x,\Omega') < |\Delta|\}$. By (6.40), the assumption $\Delta t_h \to 0$ as $h \to 0$ and the boundedness of $(u_0^h)_h$ in $L^1(\Omega'_\Delta)$, then the space translates of $\bar U^h$ on Ω' are estimated uniformly for all sequence $(h_i)_i$ convergent to zero. In the sequel, we drop the subscript i in the notation.

Proof of (6.41). Now we show a uniform estimate of the time translates of $(\bar U^h)_h$:

$$\text{for all } \Delta \in (0,\tau], \quad \int_0^{+\infty}\!\!\int_{\Omega} |U^h(t+\Delta,x) - U^h(t,x)|\,dxdt \leq \widetilde\omega(\tau) \tag{6.44}$$

uniformly in h. Here $\widetilde\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a modulus of continuity, i.e., $\lim\limits_{\tau\to 0}\widetilde\omega(\tau) = 0$.

Let us construct $\widetilde\omega(\cdot)$ verifying (6.44). First fix h and fix $\Delta \in (0,\tau]$. Denote by $I^h(\Delta)$ the left-hand side of (6.44). For $t \geq 0$, set $W^h(t,\cdot) = \bar U^h(t+\Delta,\cdot) - \bar U^h(t,\cdot)$. Notice that $W^h(t,\cdot) \equiv 0$ for large t.

Take a standard family $(\rho_\delta)_\delta$ of mollifiers on $\mathbb{R}^l$ defined as $\rho_\delta(x) := \delta^{-l}\rho(x/\delta)$, where ρ is a Lipschitz continuous, nonnegative function supported in the unit ball of $\mathbb{R}^l$, and $\int_{\mathbb{R}^l}\rho(x)\,dx = 1$. In particular, we have

$$|\nabla\rho_\delta| \leq \frac{C}{\delta^{l+1}}.$$

Here and throughout the proof, C will denote a generic constant independent of h and δ. For all $t > 0$, define the function $\varphi(t,\cdot) : \mathbb{R}^l \longrightarrow \mathbb{R}$ by $\varphi(t) := \rho_\delta * (\mathrm{sign}\,w^h(t)1\!\!1_{\Omega'})$. In order to lighten the notation, we do not stress the dependence of φ on h and δ. Discretize $\varphi(t,\cdot)$ on the mesh $\mathcal{T}_h$ by setting $\varphi_K(t) = \frac{1}{|K|}\int_K \varphi(t,x)\,dx$; we denote $\varphi^h(t)$ the corresponding discrete function. Denote $\mathrm{size}\,(\mathcal{T}_h) := \max_{K\in\mathcal{T}_h}\mathrm{diam}\,(K)$. By the definition of $\varphi(t,\cdot)$, the discrete function $\varphi^h(t)$ is null on the set $\left\{x \in \Omega \,\middle|\, \mathrm{dist}\,(x,\overline{\Omega'}) \geq \delta + \mathrm{size}\,(\mathcal{T}_h)\right\}$, for all t. Thus for all sufficiently small h and δ, the support of $\varphi^h(t)$ is included in some domain Ω'', $\Omega'' \subset \Omega$.

Note that

$$\partial_t U^h = \partial_t(\rho(p_h)s_hB(s_h)) = \partial_t(\rho(p_h)s_h)B(s_h) + \rho(p_h)s_h\partial_t B(s_h).$$

Now for for all $x \in K$, we multiply equation (6.39) by $|K|\varphi(t)_K$, integrate in t on $[s, s+\Delta]$, and make the summation over all K. Finally, we integrate the obtained equality in s over $\mathbb{R}^+$ to get

$$\begin{aligned}&\int_0^{+\infty}\sum_K |K|\varphi_K(s)W_K(s)\,ds = \\ &\int_0^{+\infty}\int_s^{s+\Delta}\sum_K |K|\,\varphi_K(t)B(s_K(t))\Big(\operatorname{div}_K[\vec{\mathcal{F}}_1^h(t)] + (f_1^h(t))_K\Big)\,dtds \\ &+\int_0^{+\infty}\int_s^{s+\Delta}\sum_K |K|\,\varphi_K(t)\rho(p_k(t))s_k(t)\beta(s_K(t))\Big(\operatorname{div}_K[\vec{\mathcal{F}}_2^h(t)] + (f_2^h(t))_K\Big)\,dtds.\end{aligned} \tag{6.45}$$

Denote by $I_\delta^h(\Delta)$ the left-hand side of (6.45). Denote $Q'' = (0,(N_h+1)\Delta t)\times\Omega''$. Using hypothesis the definitions of discrete norms and the Fubini theorem, we get

$$I_\delta^h(\Delta) \le C\Delta\left(\left\|\vec{\mathcal{F}}^h\right\|_{L^2(Q'')}(\max_{t>0}\max_{\sigma_{K,L}}|\nabla_{K,L}\varphi^h(t)| + |\nabla_{K,L}B(s_h(t))|^2) \;+\; \|\varphi^h\|_{L^\infty(Q'')}\|f^h\|_{L^1(Q'')}\right).$$

Now the the $L^2_{loc}([0,T]\times\Omega)$ bounds on $(\vec{\mathcal{F}}^h)_h$,$(f^h)_h$, the bounds $|\varphi(t,\cdot)|\le 1$ and $|\nabla\varphi(t,\cdot)|\le C/\delta^{l+1}$, yield the estimate

$$I_\delta^h(\Delta) \le C\Delta(1+\delta^{-l-1}) \tag{6.46}$$

for all h and δ small enough, uniformly in h. Now, notice that by the definition of $\varphi_K(t)$,

$$\begin{aligned}|K|\Big(|W_K(t)| - W_K(t)\varphi_K(t)\Big) &= |K|\,|W^h(t,x)| - W_K(t)\int_K\varphi(t,x)\,dx \\ &= \int_K\Big(|W^h(t,x)| - W^h(t,x)\varphi(t,x)\Big)\,dx;\end{aligned} \tag{6.47}$$

therefore

$$I^h(\Delta) - I_\delta^h(\Delta) = \int_0^{+\infty}\int_\Omega\Big(|W^h(t,x)| - W^h(t,x)\varphi(t,x)\Big)\,dxdt. \tag{6.48}$$

Starting from this point, the argument of Kruzhkov Ref. [**?**] applies exactly as for the "continuous" case. Set $U'_\delta := \Big\{x\in\mathbb{R}^l \,\Big|\, \operatorname{dist}(x,\partial\Omega') < \delta\Big\}$; notice that $U'_\delta\subset\Omega''\subset\Omega$ for all δ small enough. Notice that without loss of restriction, the boundary of Ω' can be chosen regular enough so that to ensure that $\operatorname{meas}(U'_\delta)$ goes to zero as $\delta\to 0$. By the result of Step 1 of the lemma and the Frechet-Kolmogorov theorem, the family $\Big(\int_0^{+\infty}|W^h(t,\cdot)|dt\Big)_h$ is relatively compact in $L^1_{loc}(\Omega)$. Therefore these functions are equi-integrable on Ω'', so that $\int_0^{+\infty}\!\!\int_{U'_\delta}|W^h(t,x)|dxdt \le \hat\omega(\delta)$ uniformly in h, with $\lim_{\delta\to 0}\hat\omega(\delta) = 0$. Now by the definition of φ, from formula (6.48) we deduce that

$$
\begin{aligned}
|I^h(\vartriangle) - I^h_\delta(\vartriangle)| \le 2\int_0^{+\infty}\!\!\int_{U'_\delta}&|W^h(t,x)|dxdt \\
&+ \int_0^{+\infty}\!\!\int_{\Omega'\setminus U'_\delta}\Big||W^h(t,x)| - W^h(t,x)(\rho_\delta * \operatorname{sign} W^h(t))(x)\Big|\,dxdt,
\end{aligned}
$$

the first term in the right-hand side accounts for the action of the truncation $1\!\!1_{\Omega'}$ in the definition of φ. Using the standard properties of ρ_δ, we infer

$$
\begin{aligned}
|I^h(\vartriangle) - I^h_\delta(\vartriangle)| &\le 2\hat\omega(\delta) \\
&+ \int_0^{+\infty}\!\!\int_{\Omega'\setminus U'_\delta}\int_{\mathbb{R}^l}\rho_\delta(x-y)\Big||W^h(t,x)| - W^h(t,x)\operatorname{sign} W^h(t,y)\Big|\,dydxdt.
\end{aligned}
$$

Now note the key inequality :

$$
\forall a, b \in \mathbb{R} \qquad \Big||a| - a\operatorname{sign} b\Big| \le 2\,|a-b|.
$$

Setting $\sigma := (x-y)/\delta$, we infer

$$
\begin{aligned}
|I^h(\vartriangle) - I^h_\delta(\vartriangle)| \;&\le\; 2\hat\omega(\delta) \;+\; 2\int_0^{+\infty}\!\!\int_{\Omega'}\int_{\mathbb{R}^l}\rho_\delta(x-y)|W^h(t,x) - W^h(t,y)|\,dydxdt \;\le \\
&\le\; 2\hat\omega(\delta) \;+\; 2\int_{\mathbb{R}^l}\rho(\sigma)\int_0^{+\infty}\!\!\int_{\Omega'}|\bar u^h(t,x) - \bar u^h(t,x-\delta\sigma)|\,dxdt\,d\sigma \;\le\; 2\hat\omega(\delta) + 2\omega(\delta),
\end{aligned}
\tag{6.49}
$$

where $\omega(\cdot)$ is the modulus of continuity controlling the space translates of $\bar u^h$ in Ω'. Indeed, by Steps 1 and 2 of the proof, one can choose $\omega(\cdot)$ independent of h. Combining (6.46) with (6.49), we conclude that the function

$$
\widetilde\omega(\tau) := \inf_{\delta>0} C\left\{\, \tau\,(1+\delta^{-l-1}) + 2\hat\omega(\delta) + 2\omega(\delta)\,\right\}
$$

upper bounds the quantity I^h. Because $\widetilde\omega(\tau)$ tends to 0 as $\tau \to 0$, this proves (6.44). □

7 Convergence of the finite volume scheme

Proposition 4.3. *There exists a subsequences, still denoted $(U_h, s_h)_h$, such that, as $h \to 0$*

$$\|U^h - \bar{U}^h\|_{L^1(\Omega')} \longrightarrow 0, \tag{7.50}$$
$$U_h \longrightarrow U \text{ strongly in } L^p(Q_T) \text{ and a.e. in } Q_T \text{ for all } p \geq 1, \tag{7.51}$$
$$s_h \longrightarrow s \text{ strongly in } L^p(Q_T) \text{ for all } p > 1, \tag{7.52}$$
$$\nabla_h \beta(s_h) \longrightarrow \nabla \beta(s) \text{ weakly in } (L^2(Q_T))^3, \tag{7.53}$$
$$\nabla_h p_h \longrightarrow \nabla p \text{ weakly in } (L^2(Q_T))^3, \tag{7.54}$$
$$p_h \longrightarrow p \text{ weakly in } L^2(Q_T). \tag{7.55}$$
$$\tag{7.56}$$

Furthermore,

$$s_h \longrightarrow s \text{ a.e. in } Q_T, \text{ and } 0 \leq s \leq 1 \text{ a.e. in } Q_T, \tag{7.57}$$
$$U = \phi \rho(p) s B(s) \text{ a.e. in } Q_T \tag{7.58}$$

Finally, we have,

$$f_1(p_h) f_2(s_h) \longrightarrow f_1(p) f_2(s) \text{ a.e. in } Q_T, \forall f_1, f_2 \in \mathcal{C}_b^0(\mathbb{R}) \text{ such that } f_2(0) = 0. \tag{7.59}$$

Démonstration. For the first convergence (7.50) it is useful to introduce the following inequality, for all $a, b \in \mathbb{R}$,

$$\int_0^1 |\theta a + (1-\theta) b| \, d\theta \geq \frac{1}{2}(|a| + |b|)$$

Applying this inequality to $a = u_h^{(n+1)} - u_h^n$, $b = u_h^n - u_h^{(n-1)}$, from the definition of $\bar{U}^h$ we deduce

$$\int_0^T \int_{\Omega'} |U^h(t,x) - \bar{U}^h(t,x)| \, dxdt \leq 2 \int_0^{T+\Delta t_h} \int_{\Omega'} |\bar{U}^h(t+\Delta t_h, x) - \bar{U}^h(t,x)| \, dxdt.$$

Since Δt_h tends to zero as $h \to 0$, estimate (6.44) in Lemma 4.6 implies that the right-hand side of the above inequality converges to zero as h tends to zero, and this established (7.50).

By the Riesz-Frechet-Kolmogorov compactness criterion, the relative compactness of $(\bar{U}^h)_h$ in $L^1(Q_T)$ is a consequence of Lemma 4.6. Now, the convergence (7.51) in $L^1(Q_T)$ and a.e in Q_T becomes a consequence of (7.50). Due to the fact that

U^h is bounded, we establish the convergence in $L^1(Q_T)$.
In order to prove the third convergence (7.52), we reproduce the previous lemma 4.6 for $U^h = s_h B(s_h)$, and as an application of the Riesz-Frechet-Kolmogorov compactness criterion we establish (7.52).
For the weak convergence of the discrete gradient of the global pressure, let us recall the piecewise approximation $\nabla_h p_h$ of ∇p_h in Q_t :

$$\nabla_h p_h(t,x) = \begin{cases} l\frac{P_L^n - P_K^n}{d_{K,L}}\eta_{K,L} & \text{if } (t,x) \in (t^n, t^{n+1}) \times T_{K,L}, \\ 0 & \text{if } (t,x) \in (t^n, t^{n+1}) \times T_{K,\sigma}^{\text{ext}}, \end{cases}$$

for all $K \in \mathcal{T}$ and $0 \leq n \leq N_h$. It follows from proposition 4.1 that, the sequence $(\nabla_h p_h)_h$ is bounded in $(L^2(Q_T))^3$, and as a consequence of the discrete Poincaré inequality, the sequence $(p_h)_h$ is bounded in $L^2(Q_T)$. Therefore there exist two functions $p \in L^2(Q_T)$ and $\zeta \in (L^2(Q_T))^3$ such that (7.55) holds and

$$\nabla_h p_h \longrightarrow \zeta \text{ weakly in } (L^2(Q_T))^3.$$

It remains to identify ∇p by ζ in the sense of distributions. For that, it is enough to show as $h \to 0$:

$$E_h := \int\int_{Q_T} \nabla_h p_h \cdot \varphi \, dxdt + \int\int_{Q_T} p_h \operatorname{div} \varphi \, dxdt \longrightarrow 0, \qquad \forall \varphi \in \mathcal{D}(Q_T)^3.$$

Let h be small enough such that φ vanishes in $T_{K,\sigma}^{\text{ext}}$ for all $K \in \mathcal{T}$. In view of $\eta_{K,L} = -\eta L, K$ we obtain for all $t \in (t^n, t^{n+1})$

$$\begin{aligned} \int_\Omega p_h \operatorname{div} \varphi(t,x) \, dx &= \sum_{K \in \mathcal{T}} \int_K p_h \operatorname{div} \varphi(t,x) \, dx \\ &= \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} p_K^n \int_{\sigma_{K,L}} \varphi(t,s) \cdot \eta_{K,L} \, ds \\ &= \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} (p_K^n - p_L^n) \int_{\sigma_{K,L}} \varphi(t,s) \cdot \eta_{K,L} \, ds. \end{aligned}$$

Now, from the definition of the discrete gradient,

$$\begin{aligned} \int_\Omega \nabla_h p_h \varphi(t,x) \, dx &= \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} \int_{T_{K,L}} \nabla_h p_h \varphi(t,x) \, dx \\ &= \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} \frac{l}{d_{K,L}} (p_L^n - p_K^n) \int_{T_{K,L}} \varphi(t,x) \cdot \eta_{K,L} \, dx \end{aligned}$$

Then,

$$E_h = \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} \sigma_{K,L}(p_L^n - p_K^n)\Big(\frac{1}{\sigma_{K,L}}\int_{\sigma_{K,L}} \varphi(t,s)\cdot\eta_{K,L}\,ds - \frac{1}{T_{K,L}}\int_{T_{K,L}} \varphi(t,x)\cdot\eta_{K,L}\,dx\Big)$$

Due to the smoothness of φ, one gets

$$\Big|\frac{1}{\sigma_{K,L}}\int_{\sigma_{K,L}} \varphi(t,s)\cdot\eta_{K,L}\,ds - \frac{1}{|T_{K,L}|}\int_{T_{K,L}} \varphi(t,x)\cdot\eta_{K,L}\,dx\Big| \leq C\,h,$$

and the Cauchy-Scharwz inequality with proposition 4.1

$$\begin{aligned}|E_h| &\leq Ch\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} |\sigma_{K,L}||p_K^n - p_L^n| \\ &\leq Ch\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)} |\sigma_{K,L}|d_{K,L} \\ &\leq Ch|\Omega|T.\end{aligned}$$

Now, for the identification of the limit (7.58) :
Due to the monotonicity of the function ρ, we have

$$\int_{Q_T} (\phi_h\rho(p_h)s_hB(s_h) - \phi_h\rho(v)s_hB(s_h))\,dxdt \geq 0, \quad \forall v\in L^2(Q_T),$$

this with the strong convergence (7.51) and the weak convergence (7.55) lead to,

$$\int_{Q_T} (U - \phi\rho(v)sB(s))\,dxdt \geq 0, \quad \forall v\in L^2(Q_T).$$

Finally, choose $v = p + \delta w$ with $\delta \in]0,1]$ and $w\in L^2(Q_T)$, then

$$\int_{Q_T} (U - \phi\rho(p+\delta w)sB(s))w\,dxdt \geq 0$$

letting δ goes to zero, we establish (7.58).
To conclude the a.e. convergence (7.59),on one hand, when $s_h \to s = 0$ a.e., $f_1(p_h)f_2(s_h) \to 0 = f_1(p)f_2(s)$ a.e. (since $f_2(0) = 0$ and f_1 is bounded). On the other hand, when $s_h \to s \neq 0$, in light of (7.51) we have $f_1(p_h) \to f_1(p_1)$ a.e.. Then, $f_1(p_h)f_2(s_h) \to f_1(p)f_2(s)$ since f_1, f_2 are continuous and this establish (7.59).□

Theorem 4.2. *Assume (H1)-(H5) hold. Then the functions p,s defined in proposition 4.3 constitute a weak solution of the system (2.1)-(2.2).*

Démonstration. Let T be a fixed positive constant and $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$.

• Convergence of the discrete water equation

For the discrete water equation, we multiply the equation (3.6) by $\Delta t \varphi(t^{n+1}, x_K)$ for all $K \in \mathcal{T}$ and $n \in \{0, \dots, N\}$. This yields

$$\mathfrak{C}_1^h + \mathfrak{C}_2^h + \mathfrak{C}_3^h + \mathfrak{C}_4^h + \mathfrak{C}_5^h + \mathfrak{C}_6^h = 0$$

where

$$\begin{aligned}
\mathfrak{C}_1^h &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|\, \phi_K (s_K^{n+1} - s_K^n)\varphi(t^{n+1}, x_K),\\
\mathfrak{C}_2^h &= -\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} (\beta(s_L^{n+1}) - \beta(s_K^{n+1}))\varphi(t^{n+1}, x_K),\\
\mathfrak{C}_3^h &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\varphi(t^{n+1}, x_K),\\
\mathfrak{C}_4^h &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \bigg(\rho_2 M_2(s_L^{n+1}) \sum_{L\in N(K)} \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})^+ \, d\gamma(x)\\
&\quad - \rho_2 M_2(s_K^{n+1}) \sum_{L\in N(K)} \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})^- \, d\gamma(x)\bigg)\varphi(t^{n+1}, x_K),\\
\mathfrak{C}_5^h &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} |K|\, (s_K^{n+1} - 1) f_{P,K}^{n+1} \varphi(t^{n+1}, x_K)\\
\mathfrak{C}_6^h &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} |K|\, f_{I,K}^{n+1} \varphi(t^{n+1}, x_K).
\end{aligned}$$

Performing integration by parts and keeping in mind that $\varphi(T, x_K) = 0$ for all $K \in \mathcal{T}$, we obtain

$$\begin{aligned}
\mathfrak{C}_1^h &= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|\, \phi_K s_K^{n+1} (\varphi(t^{n+1}, x_K) - \varphi(t^n, x_K)) - \sum_{K\in\mathcal{T}} |K|\, \phi_K s_K^0 \varphi(0, x_K)\\
&= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K \phi_K s_K^n \partial_t \varphi(t, x_K)\, dx\, dt - \sum_{K\in\mathcal{T}} \int_K \phi_K s_0(x) \varphi(0, x_K)\, dx\\
&=: -\mathfrak{C}_{1,1}^h - \mathfrak{C}_{1,2}^h.
\end{aligned}$$

Let us also introduce

$$\begin{aligned}
\mathfrak{C}_1^{h,*} &= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K \phi_K s_K^n \partial_t \varphi(t, x)\, dx\, dt - \int_\Omega \phi_h s_0(x) \varphi(0, x)\, dx\\
&=: -\mathfrak{C}_{1,1}^{h,*} - \mathfrak{C}_{1,2}^{h,*}.
\end{aligned}$$

Now and due to the fact that the saturation and the porosity functions are bounded, we have

$$\begin{aligned}\left|\mathfrak{C}^h_{1,1}-\mathfrak{C}^{h,*}_{1,1}\right| &= \left|\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\phi_K s_K^n\int_{t^n}^{t^{n+1}}\int_K\Big(\partial_t\varphi(t,x_K)-\partial_t\varphi(t,x)\Big)\,dx\,dt\right|\\ &\le \phi_1\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\int_{t^n}^{t^{n+1}}\int_K\Big|\partial_t\varphi(t,x_K)-\partial_t\varphi(t,x)\Big|\,dx\,dt\end{aligned}$$

using that the function φ is regular enough, we get

$$\lim_{h\to 0}\left|\mathfrak{C}^h_{1,1}-\mathfrak{C}^{h,*}_{1,1}\right| = 0. \tag{7.60}$$

Similarly

$$\begin{aligned}\mathfrak{C}^h_{1,2}-\mathfrak{C}^{h,*}_{1,2} &= \sum_{K\in\mathcal{T}}\int_K\phi_K s_0(x)(\varphi(0,x_K)-\varphi(0,x))\,dx\\ &= \int_\Omega \phi_h s_0(x)(\varphi(0,x_K)-\varphi(0,x))\,dx.\end{aligned}$$

By the regularity of φ, there exists a positive constant C such that $|\varphi(0,x_K)-\varphi(0,x)|\le C\,h$. This implies

$$\left|\mathfrak{C}^h_{1,2}-\mathfrak{C}^{h,*}_{1,2}\right| \le C\,h\phi_1\sum_{K\in\mathcal{T}}\int_K s_0(x)\,dx.$$

Sending $h\to 0$ in the above inequality, we get

$$\lim_{h\to 0}\left|\mathfrak{C}^h_{1,2}-\mathfrak{C}^{h,*}_{1,2}\right| = 0. \tag{7.61}$$

Combining (7.60) with (7.61), we obtain

$$\lim_{h\to 0}\left|\mathfrak{C}^h_{1}-\mathfrak{C}^{h,*}_{1}\right| = 0, \tag{7.62}$$

but, $\mathfrak{C}^{h,*}_1$ can be written equivalently,

$$\mathfrak{C}^{h,*}_1 = \int_{Q_T}\phi_h s_h\partial_t\varphi(t,x)\,dx\,dt - \int_\Omega \phi_h s^0\varphi(0,x)\,dx.$$

Since the bounded functions ϕ_h and s_h converge almost everywhere respectively to ϕ and s, and as a consequence of Lebesgue dominated convergence theorem, we get

$$\lim_{h\to 0}\mathfrak{C}^h_1 = \lim_{h\to 0}\mathfrak{C}^{h,*}_1 = \int_{Q_T}\phi s\partial_t\varphi(t,x)\,dx\,dt - \int_\Omega \phi s^0\varphi(0,x)\,dx.$$

Now, let us show that

$$\lim_{h\to 0} \mathfrak{C}_2^h = \int_0^T \int_\Omega \nabla\beta(s)\cdot\nabla\varphi \, dxdt. \tag{7.63}$$

Integrating by parts

$$\begin{aligned}\mathfrak{C}_2^h &= \frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} |T_{K,L}| \, l \, \frac{\beta(s_L^{n+1})-\beta(s_K^{n+1})}{d_{K,L}} \frac{\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)}{d_{K,L}} \\ &= \frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} |T_{K,L}| \, \nabla_{K,L}\beta(s_h^{n+1})\cdot\eta_{K,l}\nabla\varphi(t^{n+1},x_{K,L})\cdot\eta_{K,L},\end{aligned}$$

where $x_{K,L} = \theta x_K + (1-\theta)x_L$, $0<\theta<1$, is some point on the segment $]x_K, x_L[$. Recall that the value of $\nabla_{K,L}$ is directed by $\eta_{K,L}$, so

$$\nabla_{K,L}\beta(s_h^{n+1})\cdot\eta_{K,l}\nabla\varphi(t^{n+1},x_{K,L})\cdot\eta_{K,L} = \nabla_{K,L}\beta(s_h^{n+1})\cdot\nabla\varphi(t^{n+1},x_{K,L}).$$

since each term corresponding to the diamond $T_{K,L}$ appears twice,

$$\mathfrak{C}_2^h = \int_0^T \int_\Omega \nabla_h\beta(s_h)\cdot(\nabla\varphi)_h \, dxdt,$$

where

$$(\nabla\varphi)_h|_{(t^n,t^{n+1}]\times T_{K,L}} := \nabla\varphi(t^{n+1},x_{K,L})$$

Observe that from the continuity of φ we get $(\nabla\varphi)_h \to \nabla\varphi$ in $L^\infty(Q_T)$. Hence the convergence (7.63) is a consequence of (7.53).
Now, we show the convergence of the flux,

$$\lim_{h\to 0} \mathfrak{C}_3^h = -\int_0^T \int_\Omega M_2(s)\nabla p\cdot\nabla\varphi \, dxdt. \tag{7.64}$$

Perform integration by parts (4.19), thanks to the consistency of the fluxes, we obtain

$$\mathfrak{C}_3^h = -\frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\Big(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\Big).$$

For each couple of neighbours K, L we denote $s_{K,L}^{n+1}$ the minimum of s_K^{n+1} and s_L^{n+1} and we introduce,

$$\mathfrak{C}_3^{h,*} = -\frac{1}{2}\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{T}}\sum_{L\in N(K)} M_2(s_{K,L}^{n+1})dp_{K,L}^{n+1}\Big(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\Big).$$

Define $\overline{s}_h$ and $\underline{s}_h$ by

$$\overline{s}_h|_{(t^n,t^{n+1}]\times T_{K,L}} := \max\{s_K^{n+1}, s_L^{n+1}\}, \qquad \underline{s}_h|_{(t^n,t^{n+1}]\times T_{K,L}} := \min\{s_K^{n+1}, s_L^{n+1}\}.$$

Now, $\mathfrak{C}_3^{h,*}$ can be written under the following continues form,

$$\mathfrak{C}_3^{h,*} = -\int_0^T \int_\Omega M_2(\underline{s}_h)\nabla_h p_h \cdot (\nabla\varphi)_h \, dxdt.$$

By the monotonicity of β and thanks to the estimate (4.24), we have

$$\begin{aligned}
\int_0^T \int_\Omega |\beta(\overline{s}_h) - \beta(\underline{s}_h)|^2 \, dxdt &\leq \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} |T_{K,L}| \Big(\beta(s_L^{n+1}) - \beta(s_K^{n+1})\Big)^2 \\
&\leq Ch^2 \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} |\beta(s_L^{n+1}) - \beta(s_K^{n+1})|^2 \\
&\leq Ch^2.
\end{aligned}$$

Since β^{-1} is continues, we deduce up to a subsequence,

$$|\overline{s}_h - \underline{s}_h| \to 0 \text{ a.e. on } Q_T. \tag{7.65}$$

Moreover, we have ; $\underline{s}_h \leq s_h \leq \overline{s}_h$ and $s_h \to s$ a.e. on Q_T. Consequently, and due to the continuity of the mobility function M_2 we have $M(\underline{s}_h) \to M(s)$ a.e. on Q_T and in $L^p(Q_T)$ for $p < +\infty$. Using proposition 4.3 (7.54) and the strong convergence of $(\nabla\varphi)_h$ to $\nabla\varphi$, we obtain that

$$\lim_{h\to 0} \mathfrak{C}_3^{h,*} = -\int_0^T \int_\Omega M_2(s)\nabla p \cdot \nabla\varphi \, dxdt.$$

It remains to show that

$$\lim_{h\to 0} |\mathfrak{C}_3^h - \mathfrak{C}_3^{h,*}| = 0 \tag{7.66}$$

By the properties of the numerical flux function (3.1) we have

$$\begin{aligned}
&|G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}) - M_2(s_{K,L}^{n+1}) dp_{K,L}^{n+1}| \\
=&|G_2(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1}) - G_2(s_{K,L}^{n+1}, s_{K,L}^{n+1}; dp_{K,L}^{n+1})| \\
\leq&|dp_{K,L}^{n+1}| \quad \omega(2|s_L^{n+1} - s_K^{n+1}|).
\end{aligned}$$

Consequently,

$$|\mathfrak{C}_3^h - \mathfrak{C}_3^{h,*}| \leq \int_0^T \int_\Omega \omega(2|s_L^{n+1} - s_K^{n+1}|)\nabla_h p_h \cdot (\nabla\varphi)_h \, dxdt$$

Applying the Cauchy-Schwarz inequality, and thanks to the uniform bound on

$\nabla_h p_h$ and the convergence (7.65), we establish (7.66). Now, we treat the convergence of the gravity term

$$\lim_{h\to 0} \mathfrak{C}_4^h = -\int_0^T \int_\Omega \rho_2 M_2(s)\mathbf{g}\cdot\nabla\varphi \, dxdt. \tag{7.67}$$

Perform integration by parts (4.19),

$$\begin{aligned}
\mathfrak{C}_4^h &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} F_{2,K,L}^{n+1}\varphi(t^{n+1}, x_K) \\
&= -\frac{1}{2}\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} F_{2,K,L}^{n+1}\Big(\varphi(t^{n+1}, x_L) - \varphi(t^{n+1}, x_K)\Big)
\end{aligned}$$

We introduce,

$$\mathfrak{C}_4^{h,*} = -\frac{1}{2}\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} M_2(s_{K,L}^{n+1}) \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})\, d\gamma(x) \\
\Big(\varphi(t^{n+1}, x_L) - \varphi(t^{n+1}, x_K)\Big)$$

where $s_{K,L}^{n+1} := \min\{s_K^{n+1}, s_L^{n+1}\}$. We have

$$\mathfrak{C}_4^h - \mathfrak{C}_4^{h,*} = \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \Big(F_{2,K,L}^{n+1} - M_2(s_{K,L}^{n+1}) \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})\, d\gamma(x)\Big) \\
\Big(\varphi(t^{n+1}, x_L) - \varphi(t^{n+1}, x_K)\Big)$$

relying on the assumption (H2), that the mobility functions are Lipschitz, and β^{-1} is a Holder function, we deduce that

$$\begin{aligned}
&\Big|(F_{2,K,L}^{n+1} - M_2(s_{K,L}^{n+1}) \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})\, d\gamma(x)\Big| \\
&= \Big|\Big(M_2(s_L^{n+1}) - M_2(s_{K,L}^{n+1})\Big) \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})^+ \, d\gamma(x) \\
&- \Big(M_2(s_K^{n+1}) - M_2(s_{K,L}^{n+1})\Big) \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})^- \, d\gamma(x)\Big| \\
&\le C|\mathbf{g}||\sigma_{K,L}|\Big|s_L^{n+1} - s_K^{n+1}\Big| \\
&\le C|\mathbf{g}||\sigma_{K,L}|\Big|\beta(s_L^{n+1}) - \beta(s_K^{n+1})\Big|^\theta,
\end{aligned}$$

this yields to

$$|\mathfrak{C}_4^h - \mathfrak{C}_4^{h,*}| \leq C \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} |\sigma_{K,L}| d_{K,L} \Big|\beta(s_L^{n+1}) - \beta(s_K^{n+1})\Big|^{\theta} \Big|\nabla_h \varphi_h\Big|.$$

Using estimate (4.24), and Cauchy Schwarz inequality, then

$$|\mathfrak{C}_4^h - \mathfrak{C}_4^{h,*}| \to 0 \text{ when } h \to 0.$$

For the convergence of the source terms, $\mathfrak{C}_5^h + \mathfrak{C}_6^h$ can be written equivalently,

$$\begin{aligned}\mathfrak{C}_5^h + \mathfrak{C}_6^h = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K (s_K^{n+1} - 1) f_P(t,x) \varphi(t^{n+1}, x_K)\, dx dt \\ + \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K f_I(t,x) \varphi(t^{n+1}, x_K)\, dx dt\end{aligned}$$

Now, we introduce

$$\begin{aligned}\mathfrak{C}_5^{h,*} + \mathfrak{C}_6^{h,*} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K (s_K^{n+1} - 1) f_P(t,x) \varphi(t,x)\, dx dt \\ + \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K f_I(t,x) \varphi(t,x)\, dx dt\end{aligned}$$

Due to regularity of φ we obtain,

$$\Big|\mathfrak{C}_5^h + \mathfrak{C}_6^h - \mathfrak{C}_5^{h,*} - \mathfrak{C}_6^{h,*}\Big| \leq C(\Delta t + h)(\| f_P \|_{L^1(Q_T)} + \| f_I \|_{L^1(Q_T)})$$

and this ensures,

$$\lim_{h\to 0} \Big|\mathfrak{C}_5^h + \mathfrak{C}_6^h - \mathfrak{C}_5^{h,*} - \mathfrak{C}_6^{h,*}\Big| = 0.$$

We can write equivalently,

$$\mathfrak{C}_5^{h,*} + \mathfrak{C}_6^{h,*} = \int_{Q_T} (s_h - 1) f_P(t,x) \varphi(t,x)\, dx dt + \int_{Q_T} f_I(t,x) \varphi(t,x)\, dx dt$$

Finally, by the convergence of the saturation function we get,

$$\begin{aligned}\lim_{h\to 0} (\mathfrak{C}_5^h + \mathfrak{C}_6^h) &= \lim_{h\to 0} (\mathfrak{C}_5^{h,*} + \mathfrak{C}_6^{h,*}) \\ &= \int_{Q_T} (s-1) f_P(t,x) \varphi(t,x)\, dx dt + \int_{Q_T} f_I(t,x) \varphi(t,x)\, dx dt\end{aligned}$$

- Convergence of the discrete gas equation

Multiplying the discrete gas equation (3.5) by $\Delta t\varphi(t^{n+1}, x_K)$ for all $K \in \mathcal{T}$ and $n \in \{0, \dots, N\}$. Summing the result over K and n yields

$$\mathcal{C}_1^h + \mathcal{C}_2^h + \mathcal{C}_3^h + \mathcal{C}_4^h + \mathcal{C}_5^h = 0,$$

where

$$\begin{aligned}
\mathcal{C}_1^h &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|\, \phi_K(\rho(p_K^{n+1})s_K^{n+1} - \rho(p_K^n)s_K^n)\varphi(t^{n+1}, x_K),\\
\mathcal{C}_2^h &= -\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \frac{|\sigma_{K,L}|}{d_{K,L}} \rho_{K,L}^{n+1}(\beta(s_L^{n+1}) - \beta(s_K^{n+1}))\varphi(t^{n+1}, x_K),\\
\mathcal{C}_3^h &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \sum_{L\in N(K)} \rho_{K,L}^{n+1} G_1(s_K^{n+1}, s_L^{n+1}; dp_{K,L}^{n+1})\varphi(t^{n+1}, x_K),\\
\mathcal{C}_4^h &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} \Big(\rho^2(p_K^{n+1})M_1(s_K^{n+1}) \sum_{L\in N(K)} \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})^+ \, d\gamma(x)\\
&\quad - \rho^2(p_L^{n+1})M_1(s_L^{n+1}) \sum_{L\in N(K)} \int_{K/L} (\mathbf{g}\cdot\eta_{K,L})^- \, d\gamma(x)\Big)\varphi(t^{n+1}, x_K),\\
\mathcal{C}_5^h &= \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{T}} |K|\, \rho(p_K^{n+1}) s_K^{n+1} f_{P,K}^{n+1} \varphi(t^{n+1}, x_K).
\end{aligned}$$

Performing integration by parts and keeping in mind that $\varphi(T, x_K) = 0$ for all $K \in \mathcal{T}$, we obtain

$$\begin{aligned}
\mathcal{C}_1^h &= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} |K|\, \phi_K \rho(p_K^{n+1}) s_K^{n+1} (\varphi(t^{n+1}, x_K) - \varphi(t^n, x_K))\\
&\quad - \sum_{K\in\mathcal{T}} |K|\, \phi_K \rho(p_K^0) s_K^0 \varphi(0, x_K)\\
&= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} \int_{t^n}^{t^{n+1}} \int_K \phi_K \rho(p_K^{n+1}) s_K^{n+1} \partial_t \varphi(t, x_K)\, dx\, dt\\
&\quad - \sum_{K\in\mathcal{T}} \int_K \phi_K \rho(p_K^0) s_K^0 \varphi(0, x_K)\, dx\\
&=: -\mathcal{C}_{1,1}^h - \mathcal{C}_{1,2}^h.
\end{aligned}$$

Let us also introduce

$$\begin{aligned}\mathcal{C}_1^{h,*} &= -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\int_{t^n}^{t^{n+1}}\int_K \phi_K\rho(p_K^{n+1})s_K^{n+1}\partial_t\varphi(t,x)\,dx\,dt\\ &\quad -\sum_{K\in\mathcal{T}}\int_K \phi_K\rho(p_K^0)s_K^0\varphi(0,x)\,dx\\ &=: -\mathcal{C}_{1,1}^{h,*}-\mathcal{C}_{1,2}^{h,*}.\end{aligned}$$

Then

$$\mathcal{C}_{1,2}^{h}-\mathcal{C}_{1,2}^{h,*} = \sum_{K\in\mathcal{T}}\int_K \phi_K\rho(p_K^0)s_K^0(\varphi(0,x_K)-\varphi(0,x))\,dx.$$

By the regularity of φ, there exists a positive constant C such that $|\varphi(0,x_K)-\varphi(0,x)|\le C\,h$. This implies

$$\left|\mathcal{C}_{1,2}^{h}-\mathcal{C}_{1,2}^{h,*}\right|\le C\,h\,|\Omega|\,.$$

Sending $h\to 0$ in the above inequality, we get

$$\lim_{h\to 0}\left|\mathcal{C}_{1,2}^{h}-\mathcal{C}_{1,2}^{h,*}\right| = 0. \tag{7.68}$$

Now, due to the fact that the saturation function is bounded and the assumptions (H1),(H5)on the porosity and the density, we have

$$\begin{aligned}\left|\mathcal{C}_{1,1}^{h}-\mathcal{C}_{1,1}^{h,*}\right| &= \left|\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\phi_K\rho(p_K^{n+1})s_K^{n+1}\int_{t^n}^{t^{n+1}}\int_K\Big(\partial\varphi(t,x_K)-\partial\varphi(t,x)\Big)\,dx\,dt\right|\\ &\le \phi_1\rho_M\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\int_{t^n}^{t^{n+1}}\int_K\Big|\partial\varphi(t,x_K)-\partial\varphi(t,x)\Big|\,dx\,dt\end{aligned}$$

using that the function φ is regular enough, we get

$$\lim_{h\to 0}\left|\mathcal{C}_{1,1}^{h}-\mathcal{C}_{1,1}^{h,*}\right| = 0. \tag{7.69}$$

Combining (7.69) with (7.68), we obtain

$$\lim_{h\to 0}\left|\mathcal{C}_{1}^{h}-\mathcal{C}_{1}^{h,*}\right| = 0 \tag{7.70}$$

but, $\mathcal{C}_1^{h,*}$ can be written equivalently,

$$\mathcal{C}_1^{h,*} = \int_{Q_T}\phi_h\rho(p_h)s_h\partial_t\varphi(t,x)\,dx\,dt-\int_\Omega\phi_h\rho(p_h^0)s_h^0\varphi(0,x)\,dx.$$

Since $\phi_h\rho(p_h)s_h$ and $\phi_h\rho(p_h^0)s_h^0$ converge almost everywhere respectively to $\phi\rho(p)s$

and $\phi\rho(p^0)s^0$, and as a consequence of Lebesgue dominated convergence theorem, we get

$$\lim_{h\to 0}\mathcal{C}_1^h = \lim_{h\to 0}\mathcal{C}_1^{h,*} = \int_{Q_T} \phi\rho(p)s\partial_t\varphi(t,x)\,dx\,dt - \int_\Omega \phi\rho(p^0)s^0\varphi(0,x)\,dx.$$

Now, let us show that

$$\lim_{h\to 0}\mathcal{C}_2^h = \int_0^T\int_\Omega \rho(p)\nabla\beta(s)\cdot\nabla\varphi\;dxdt. \tag{7.71}$$

Integrating by parts

$$\begin{aligned}
\mathcal{C}_2^h &= \frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K,L}|\,\rho_{K,L}^{n+1}l\frac{\beta(s_L^{n+1})-\beta(s_K^{n+1})}{d_{K,L}}\frac{\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)}{d_{K,L}}\\
&= \frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}|T_{K,L}|\,\rho_{K,L}^{n+1}\nabla_{K,L}\beta(s_h^{n+1})\cdot\eta_{K,l}\nabla\varphi(t^{n+1},x_{K,L})\cdot\eta_{K,L},
\end{aligned}$$

where $x_{K,L}=\theta x_K+(1-\theta)x_L$, $0<\theta<1$, is some point on the segment $]x_K,x_L[$. Recall that the value of $\nabla_{K,L}$ is directed by $\eta_{K,L}$, so

$$\nabla_{K,L}\beta(s_h^{n+1})\cdot\eta_{K,l}\nabla\varphi(t^{n+1},x_{K,L})\cdot\eta_{K,L} = \nabla_{K,L}\beta(s_h^{n+1})\cdot\nabla\varphi(t^{n+1},x_{K,L}).$$

since each term corresponding to the diamond $T_{K,L}$ appears twice,

$$\mathcal{C}_2^h = \int_0^T\int_\Omega \rho(p_h)\nabla_h\beta(s_h)\cdot(\nabla\varphi)_h\;dxdt, \tag{7.72}$$

where

$$(\nabla\varphi)_h|_{(t^n,t^{n+1}]\times T_{K,L}} := \nabla\varphi(t^{n+1},x_{K,L})$$

Define

$$D_2^h = -\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{T}}\sum_{L\in N(K)}\frac{|\sigma_{K,L}|}{d_{K,L}}\overline{\rho}_{K,L}^{n+1}(\beta(s_L^{n+1})-\beta(s_K^{n+1}))\varphi(t^n,x_K), \tag{7.73}$$

where $\overline{\rho}_{K,L}^{n+1}=(\rho(P_K^{n+1})+\rho(P_L^{n+1})/2$. We have :

$$\begin{aligned}
(\beta(s_L^{n+1})-\beta(s_K^{n+1}))\overline{\rho}_{K,L}^{n+1} &= (\beta(s_L^{n+1})\rho(P_L^{n+1})-\beta(s_K^{n+1})\rho(P_K^{n+1}))\\
&\quad+\beta(s_L^{n+1})(\overline{\rho}_{K,L}^{n+1}-\rho(P_L^{n+1}))-\beta(s_K^{n+1})(\overline{\rho}_{K,L}^{n+1}-\rho(P_K^{n+1}))\\
&= (\beta(s_L^{n+1})\rho(P_L^{n+1})-\beta(s_K^{n+1})\rho(P_K^{n+1}))\\
&\quad+(\beta(s_L^{n+1})+\beta(s_K^{n+1}))(\rho(P_K^{n+1})-\rho(P_L^{n+1}))/2.
\end{aligned}$$

Then, D_2^h can be rewritten

$$D_2^h = D_3^h + D_4^h$$

where

$$D_3^h = \sum_{n=0}^{N-1} \Delta t \sum_{\sigma_{K,L} \in E} \tau_{K,L} \Big(\beta(s_L^{n+1})\rho(p_L^{n+1}) - \beta(s_K^{n+1})\rho(p_K^{n+1}) \Big) \Big(\varphi(t^{n+1}, x_K) - \varphi(t^{n+1}, x_L) \Big)$$

$$D_4^h = \sum_{n=0}^{N-1} \Delta t \sum_{\sigma_{K,L} \in E} \tau_{K,L} \beta_{K,L}^{n+1} \Big(\rho(p_K^{n+1}) - \rho(p_L^{n+1}) \Big) \Big(\varphi(t^{n+1}, x_K) - \varphi(t^{n+1}, x_L) \Big)$$

where $\beta_{K,L}^{n+1} = (\beta(s_L^{n+1}) + \beta(s_K^{n+1}))/2$, recall that $\tau_{K,L} = \frac{|\sigma_{K,L}|}{d_{K,L}}$. Follow the same lines as in (7.72),

$$D_3^h = \int_0^T \int_\Omega \nabla_h(\rho(s_h)\beta(s_h)) \cdot (\nabla\varphi)_h \, dxdt,$$
$$D_4^h = \int_0^T \int_\Omega \overline{\beta}(s_h)\nabla_h \rho(p_h) \cdot (\nabla\varphi)_h \, dxdt$$

Using (4.23) and (4.24), we have

$$\nabla_h(\rho(p_h)\beta(s_h)) \longrightarrow \nabla(\rho(s)\beta(s)) \text{ weakly } inL^2(Q_T),$$

and using the fact that $(\nabla\varphi)_h$ converges strongly in $L^2(Q_T)$, we have

$$D_3^h \longrightarrow \int_0^T \int_\Omega \nabla(\rho(s)\beta(s)) \cdot \nabla\varphi \, dxdt.$$

In order to handle the convergence of D_4^h we are going to show

$$\overline{\beta}(s_h) \longrightarrow \beta(s) \text{ strongly in } inL^2(Q_T), \tag{7.74}$$

and

$$\nabla_h \rho(p_h) \longrightarrow \nabla\rho^\star \text{ weakly in } inL^2(Q_T). \tag{7.75}$$

The sequence $(\rho(p_h))_h$ is bounded then

$$\rho(p_h) \longrightarrow \rho^\star \text{ weakly in } L^2(Q_T). \tag{7.76}$$

Using the fact that $\rho'(.)$ is bounded, we have

$$\begin{aligned}\|\nabla_h\rho(p_h)\|^2_{L^2(Q_T)} &= \sum_{n=0}^{N-1}\Delta t\sum_{\sigma_{K,L}\in E}\tau_{K,L}|\rho(p_L^{n+1}))-\rho(p_K^{n+1}))|^2\\ &\le \sum_{n=0}^{N-1}\Delta t\sum_{\sigma_{K,L}\in E}\tau_{K,L}|\rho'(p_{K,L})(p_L^{n+1}-p_K^{n+1})|^2,\end{aligned}$$

and using the estimate (7.54), we deduce that $\nabla_h\rho(p_h)$ is bounded in $L^2(Q_T)$ and converges weakly to a function ξ in $L^2(\Omega)$; and from (7.76) we deduce that $\xi = \nabla\rho^\star$ weakly.
Recall that

$$\beta(s_h) = \sum_{n=0}^{N-1}\sum_{K}\beta(s_K^{n+1})1_{K\times]t^n,t^{n+1}]} \longrightarrow \beta(s) \text{ strongly in } L^2(Q_T).$$

Let us show for $\overline{\beta}(s_h) = \sum_{n=0}^{N-1}\sum_{\sigma_{K,L}\in E}\beta_{K,L}^{n+1}1_{T_{K,L}\times]t^n,t^{n+1}]}$,
where $\beta_{K,L}^{n+1} = \frac{\beta(s_L^{n+1})+\beta(s_K^{n+1})}{2}$, that

$$\overline{\beta}(s_h) - \beta(s_h) \longrightarrow 0 \text{ strongly in } L^2(Q_T)$$

In fact,

$$\begin{aligned}&\|\overline{\beta}(s_h)-\beta(s_h)\|^2_{L^2(Q_T)}\\ &= \sum_{n=0}^{N-1}\Delta t\sum_{\sigma_{K,L}\in E}\Big(|T_{K,L}\cap K||\beta_{K,L}^{n+1}-\beta_K^{n+1}|^2+|T_{K,L}\cap L||\beta_{K,L}^{n+1}-\beta_L^{n+1}|^2\Big)\\ &\le \frac{1}{2}\sum_{n=0}^{N-1}\Delta t\sum_{\sigma_{K,L}\in E}|T_{K,L}|(|\beta_K^{n+1}-\beta_L^{n+1}|^2),\end{aligned}\tag{7.77}$$

and from estimate (4.24), there exists a positive constant C such that

$$|\overline{\beta}(s_h)-\beta(s_h)|^2_{L^2(Q_T)} \le Ch^2$$

which establish the desired limit. Then, the convergences (7.75), leads to

$$\lim_{h\to 0} D_4^h = \int_0^T\int_\Omega \beta(s)\nabla\rho^\star\cdot\nabla\varphi\, dxdt. \tag{7.78}$$

Finally, let us show $C_2^h - D_2^h \to 0$. We have

$$C_2^h - D_2^h = \sum_{n=0}^{N-1}\Delta t\sum_{\sigma_{K,L}}\tau_K, L(\tilde{\rho}_{K,L}^{n+1}\Big(\beta(s_L^{n+1})-\beta(s_K^{n+1})\Big)\Big(\varphi(t^{n+1},x_L)-\varphi(t^{n+1},x_K)\Big),$$

where $\tilde{\rho}_{K,L}^{n+1} = \rho_{K,L}^{n+1} - \overline{\rho}_{K,L}^{n+1}$. This expression can be rewritten as

$$C_2^h - D_2^h = \int_{Q_T} \tilde{\rho}(p_h)\nabla_h \beta(s_h) \cdot (\nabla\phi)_h \, dxdt \tag{7.79}$$

Let us show that $\tilde{\rho}(p_h) \to 0$ strongly in $L^2(Q_T)$.
We have

$$\tilde{\rho}_{K,L}^{n+1} = \rho_{K,L}^{n+1} - \overline{\rho}_{K,L}^{n+1} = \frac{1}{p_L^{n+1} - p_K^{n+1}} \int_{p_K^{n+1}}^{p_L^{n+1}} \rho(\psi)\, d\psi - \frac{\rho(p_K^{n+1}) + \rho(p_L^{n+1})}{2},$$

and from hypothesis (H5), the function ρ is monotone and uniformly Lipschitz, then there exists a positive such that

$$\tilde{\rho}_{K,L}^{n+1} \leq C|p_L^{n+1} - p_K^{n+1}|.$$

So,

$$\|\tilde{\rho}(p_h)\|_{L^2(Q_T)}^2 = \sum_{n=0}^{N-1} \Delta t \sum_{\sigma_{K,L} \in E} |T_{K,L}||\tilde{\rho}_{K,L}^{n+1}|^2 \tag{7.80}$$

$$\leq C \sum_{n=0}^{N-1} \Delta t \sum_{\sigma_{K,L} \in E} |T_{K,L}|(|p_L^{n+1} - p_K^{n+1}|^2), \tag{7.81}$$

using (4.23), we deduce

$$\|\tilde{\rho}(p_h)\|_{L^2(Q_T)}^2 \leq Ch^2,$$

which goes to zero when h goes to zero. This convergence combined with the weak convergence (7.54) and the strong convergence of $(\nabla\phi)_h$ in $L^\infty(Q_T)$ shows that

$$C_2^h - D_2^h \longrightarrow 0, \text{ when } h \to 0.$$

□

Remark 4.3. *In the case where $\mathbf{K}(x)$, the permeability tensor of the porous medium at a point x, considered to be*

$$\mathbf{K}(x) = k(x)\mathcal{I}_d$$

where k is a scalar bounded function of the space, $k(x) \geq k_0 > 0$ and $\mathcal{I}_d$ is the identity matrix. The main part is the approximation of the dissipative terms (capillary terms) on each interface $\sigma_{K,L}$ as follows :

Denote by,

$$k_K = \frac{1}{|K|}\int_K k(x)\,dx.$$

Now, we consider the following approximation

$$\int_{\sigma_{K,L}} k(x)\rho(p)\nabla\beta(s)\cdot\eta_{K,L}\,d\gamma \approx d^*_{K,L}\rho_{K,L}\frac{|\sigma_{K,L}|}{d_{K,L}}(\beta(s_L)-\beta(s_K))$$

$$\int_{\sigma_{K,L}} k(x)\rho(p)M_1(s)\nabla p\cdot\eta_K\,d\gamma \approx d^*_{K,L}\rho_{K,L}\Big(-M_1(s_L)(dp_{K,L})^+ + M_1(s_K)(dp_{K,L})^-\Big)$$

$$\int_{\sigma_{K,L}} k(x)\rho^2(p)M_1(s)\mathbf{g}\cdot\eta_K\,d\gamma \approx d^*_{K,L}\Big(\rho^2(p_K^{n+1})M_1(s_K^{n+1})\mathbf{g}_{K,L} - \rho^2(p_L^{n+1})M_1(s_L^{n+1})\mathbf{g}_{L,K}\Big)$$

where,

$$d^*_{K,L} = \frac{k_{K,L}\,k_{L,K}}{d(x_K,\sigma_{K,L})k_{K,L} + d(x_L,\sigma_{K,L})k_{L,K}}\,d(x_K,x_L), \tag{7.82}$$

$$dp_{K,L} = \frac{|\sigma_{K,L}|}{d_{K,L}}\Big(p_L - p_K\Big) = (dp_{K,L})^+ - (dp_{K,L})^-,$$

$$\mathbf{g}_{K,L} := \int_{K/L}(\mathbf{g}\cdot\eta_{K,L})^+\,d\sigma = \int_{K/L}(\mathbf{g}\cdot\eta_{L,K})^-\,d\gamma(x)$$

and,

$$\rho_{K,L} = \begin{cases} \dfrac{1}{p_L - p_K}\displaystyle\int_{p_K}^{p_L}\rho(\xi)\,d\xi & \textit{if } p_L - p_K \neq 0 \\ \rho(p_K) & \textit{otherwise} \end{cases}$$

In (7.82), we take the harmonic average on the interfaces in order to ensure the conservation of numerical fluxes.

CHAPITRE 5

Conclusion

The progress done during the preparation of this thesis, is a contribution for some difficult non linear degenerate strongly coupled systems. The progress contains two major themes.

Firstly, a theoretical study of(existence of solutions) :
-Two compressible immiscible flow in porous media : The case where the density of each fluid depends on its corresponding pressure.
-Compressible/incompressible (gas/water) flow in porous media : The case where the density of gas depends on its corresponding pressure.

Secondly, construction and convergence analysis of finite volume schemes :
-Construction and convergence analysis of a finite volume discretization for compressible/incompressible (gas/water) flow in a multi-dimensional porous media : The case where the density depends on global pressure.

The subject still of interest and of current events from both numerical and theoretical themes. Under the context of the first theme (theoretical study),compressible immiscible multi-phase flow model still an open problem. We implement the same technics done in this thesis for the formulation under the "total differential condition" and we are not far away from obtaining an existence result for this model. For the second theme (FV), indeed, it is possible to construct finite volume discretizations for two (and multi-) phase compressible immiscible flow model, the construction and the convergence of such discretization is a theme of future papers.

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